Multivariate asset models using Lévy processes and applications

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In this paper we propose a multivariate asset model based on Lévy processes for pricing of products written on more than one underlying asset. Our construction is based on a two factor representation of the dynamics of the asset log-returns. We investigate the properties of the model and introduce a multivariate generalization of some processes which are quite common in financial applications, such as subordinated Brownian motions, jump diffusion processes and time changed Lévy processes. Finally, we explore the issue of model calibration for the proposed setting and illustrate its robustness on a number of numerical examples.

Keywords: Jump Diffusion process, Lévy processes, model calibration, multinames derivative contracts, subordinated Brownian motions, time changed Lévy processes.

Date: May 21, 2012

1. Introduction

The aim of this paper is to introduce a simple, parsimonious and robust model for multivariate Lévy processes with dependence between components, which can be easily implemented for financial applications, such as the pricing of several types of basket options commonly used in the credit and the energy derivatives markets.

The proposed model generalizes the approaches existing in the literature to any class of multivariate Lévy process, from subordinated Brownian motions to jump diffusion models. Further, the model has a simple and intuitive economic interpretation. Our construction, in fact, is based on a parsimonious two-factor linear representation of the assets (log)-returns, in the sense that it uses a linear combination of two independent Lévy processes representing respectively the systematic factor and the idiosyncratic shock. Consequently, dependence between assets in a given portfolio is originated by the common component of the overall risk. As Lévy processes are invariant under linear transformations, our approach allows to specify any one-dimensional model for each of the components. In order to guarantee that the model still allows a calibration procedure to market prices of traded vanilla options, which is parsimonious and independent of the factor structure adopted for the asset (log-)returns, in this paper we also study the conditions under which dependence can be separated from the behaviour of the margins. The construction put forward in this

paper can be further applied to originate multidimensional versions of time changed Lévy processes with dependence between components. This would allow to incorporate stochastic volatility features which are shown to improve the performance of Lévy processes in pricing options across different maturities (see, e.g., Carr et al., 2003, Carr and Wu, 2004 and Huang and Wu, 2004).

As the recent crisis in the financial markets has stressed even more the importance of capturing market shocks with more refined models (compared to Brownian motions, i.e. Gaussian distribution), in the last few years, attention has been paid to the construction of multidimensional financial models based on Lévy processes, with applications especially in the credit derivatives area; the proliferation of complex multivariate products, such as CDOs, has in fact highlighted the need of simple and parsimonious models describing both dependence among the underlying instruments, and changes in their creditworthiness induced by shocks (jumps).

The idea of inducing correlation via a factor approach dates back to Vasicek (1987) for the case of Brownian motions; the application of linear transformations has been extensively adopted in the literature for the case of Lévy processes as well. For example, Baxter (2007) and Moosbrucker (2006.a, 2006.b) use a factor copula approach for both time changed Brownian motions, such as the Variance Gamma (VG) process and the Normal Inverse Gaussian process (NIG), and Jump Diffusion (JD) processes of the Merton (Merton, 1976) and the Kou (Kou, 2002) type. The common traits of these contributions are the choice of independent copies of the same stochastic processes to model the margin, the systematic and the idiosyncratic processes, the same pairwise correlation coefficient for all margin processes, and the fact that this correlation coefficient can only be positive due to restrictions on the model parameters required to guarantee the existence of the corresponding characteristic functions. More complex and richer models have been put forward by Lindskog and McNeil (2003) for the case of Poisson processes and Semeraro (2008) and Luciano and Semeraro (2010) for subordinated Brownian motions. In details, Lindskog and McNeil (2003) make use of linear combinations to develop a common Poisson shock process framework to model dependent events frequencies in the context of insurance loss modelling and credit risk modelling. This approach is then extended in Brigo et al. (2007) to a formulation which avoids repeated defaults at both cluster level and single name level. Semeraro (2008) and Luciano and Semeraro (2010) generalize the approaches of Luciano and Schoutens (2006), Cont and Tankov (2004) and Eberlein and Madan (2010), by introducing a multivariate subordinator incorporating both global and idiosyncratic components. This construction recognizes that business time is related to trading activity with both global and idiosyncratic parts, as supported by the empirical evidence put forward by Lo and Wang (2000). To further improve the richness of the correlation structure, Semeraro (2008) and Luciano and Semeraro (2010) also use correlated Brownian motions; as pointed out by the authors, though, this general formulation is no longer parsimonious in terms of parameters:

the presence of a correlation matrix for the Brownian motion part of the components, in fact, implies that the number of parameters grows with the square of the number of assets included in the basket. The same dimensionality problem is suffered by the approach put forward by Kawai (2009), in which the number of the components is the same as the number of the names in the portfolio; further, the construction applies only to pure jump processes with the same variance.

Although similar in principle to the constructions put forward by Moosbrucker (2006.a, 2006.b), Luciano and Semeraro (2010) and Brigo et al. (2007), our model presents some distinctive features. In first place, it allows a richer correlation structure (including negative linear correlation), which is also facilitated by the fact that the idiosyncratic shock and the systematic component can be chosen separately one of the other. Further, our construction extends from the pure jump setting to accommodate any type of Lévy process; in the case of jump diffusion processes, it allows the distribution of the jump sizes to depend on the nature of the underlying shock; in the case of subordinated Brownian motions, instead, the construction does not necessarily rely on the law/characteristic function of the process chosen as subordinator. This is of relevance, for example, in those cases in which the simulation of the process subordinator proves inefficient, as in the case of the CGMY process (see, e.g., Ballotta and Kyriacou, 2011). Finally, the proposed model is relatively parsimonious in terms of the overall number of parameters involved, as this grows linearly with the number of assets, which facilitates its calibration to market data.

The remaining of the paper is organized as follows. In section 2, we introduce the construction for multivariate Lévy processes, investigate the properties of the resulting model and show how, under certain conditions, the proposed formulation can be used to build multivariate subordinated Brownian motions and jump diffusion processes. A financial application aimed at discussing the issue of model calibration and testing the robustness and the flexibility of the model is presented in section 3; in this section, we also consider the pricing of spread options and illustrate a potential application of the proposed model for recovering information on the implied correlation matrix. Extensions to the case of time changed Lévy processes are introduced in section 4, whilst section 5 concludes.

2. Multivariate Lévy process via linear transformation

Lévy processes are characterized by independent and stationary increments; they are fully described by their characteristic function which admits Lévy-Khintchine representation

$$\begin{aligned}\phi(u;t) &= e^{t\varphi(u)}, \qquad u \in \mathbb{R}\\ \varphi(u) &= iu\alpha - u^2\frac{\sigma^2}{2} + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux1_{(|x|<1)}\right)\Pi(dx).\end{aligned}$$

The terms in the characteristic exponent, $\varphi(\cdot)$, i.e. (α, σ, Π) represent the characteristic triple of the Lévy process. The parameter $\alpha \in \mathbb{R}$ describes the drift of the process, $\sigma > 0$ represents its diffusion part, whilst the jumps are fully characterized by the Lévy measure Π, i.e. a positive measure satisfying

$$\int_{\mathbb{R}} \left(1 \wedge |x|^2\right) \Pi(dx) < \infty.$$

2.1. General framework

To construct a multivariate Lévy process with dependent components, we use the property that these processes are invariant under linear transformations. The main result is given in the following.

Proposition 1. *Let*

$$\mathbf{Y}(t) = \begin{pmatrix} Y_1(t) \\ \vdots \\ Y_n(t) \end{pmatrix},$$

and $Z(t)$ be independent Lévy processes on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with characteristic functions $\phi_{Y_j}(u;t)$, for $j = 1, ..., n$, and $\phi_Z(u;t)$ respectively. Then, for $a_j \in \mathbb{R}, j = 1, ..., n$

$$\mathbf{X}(t) = \begin{pmatrix} X_1(t) \\ \vdots \\ X_n(t) \end{pmatrix} = \begin{pmatrix} Y_1(t) + a_1 Z(t) \\ \vdots \\ Y_n(t) + a_n Z(t) \end{pmatrix}$$

is a Lévy process on $\mathbb{R}^n$ with characteristic function

$$\phi_{\mathbf{X}}(\mathbf{u};t) = \phi_Z\left(\sum_{j=1}^{n} a_j u_j; t\right) \prod_{j=1}^{n} \phi_{Y_j}(u_j;t), \quad \mathbf{u} \in \mathbb{R}^n. \tag{2.1}$$

The proof follows from the properties of a Lévy process (see, for example, Theorem 4.1 in Cont and Tankov, 2004).

The following holds by construction (full proof in Appendix A.1).

Corollary 1. *Let $\mathbf{X}(t)$ be the multivariate Lévy process introduced in Proposition 1. Then.*

(i) For $j = 1, ..., n$, the mth cumulant, c_m, of the jth component of $\mathbf{X}(t)$ is

$$c_m(X_j(t)) = t\left[c_m(Y_j(1)) + a_j^m c_m(Z(1))\right]. \tag{2.2}$$

(ii) For any $j \neq l$, the covariance between the jth and l^{th} components of $\mathbf{X}(t)$ is

$$\mathbb{C}ov(X_j(t), X_l(t)) = a_j a_l \mathbb{V}ar(Z(1))\, t.$$

The construction given in Proposition 1 offers a simple and intuitive economic interpretation as for each margin, X_j, the process Z can be considered as the systematic part of the risk, whilst the process Y_j can be seen as capturing the idiosyncratic shock. Due to the presence of the common factor $Z(t)$, the components of $\mathbf{X}(t)$ may jump together, inducing non linear dependence as described in the following (the proof is presented in Appendix A.2).

Corollary 2. *For each $t \geq 0$, the multivariate Lévy process $\mathbf{X}(t)$ is positive associated if either $a_j > 0$ for $j = 1, ..., n$ or $a_j < 0$ for $j = 1, ..., n$, i.e.*

$$\mathbb{C}ov\left(f\left(\mathbf{X}(t)\right), g\left(\mathbf{X}(t)\right)\right) \geq 0$$

for all increasing function $f, g : \mathbb{R}^n \to \mathbb{R}$ for which the covariance is well-defined.

In the more general case in which the coefficients a_j do not have the same sign for $j = 1, ..., n$, the components $X_j(t)$ and $X_l(t)$, $j \neq l$, are pairwise negative quadrant dependent if $a_j a_l < 0$. Moreover, it follows directly from the construction of $\mathbf{X}(t)$ that if $\mathbf{Y}(t)$ is degenerate, the components of $\mathbf{X}(t)$ are perfectly (linear) dependent; on the other hand, if $Z(t)$ is degenerate, the components of $\mathbf{X}(t)$ are independent.

For the case of the proposed construction, the dependence between components of the multivariate Lévy process $\mathbf{X}(t)$ is correctly described by the pairwise linear correlation coefficient

$$\rho_{jl}^{\mathbf{X}} = \mathbb{C}orr\left(X_j(t), X_l(t)\right) = \frac{a_j a_l \mathbb{V}ar\left(Z(1)\right)}{\sqrt{\mathbb{V}ar\left(X_j(1)\right)}\sqrt{\mathbb{V}ar\left(X_l(1)\right)}}. \tag{2.3}$$

Equation (2.2) implies that $-1 \leq \rho_{jl}^{\mathbf{X}} \leq 1$; in more details, $\rho_{jl}^{\mathbf{X}} = 0$ if and only if either $a_j a_l = 0$ or $\mathbb{V}ar\left(Z(1)\right) = 0$, i.e. Z is degenerate and the margins are independent. Further, $|\rho_{jl}^{\mathbf{X}}| = 1$ if and only if $\mathbf{Y}(t)$ is degenerate and there is no idiosyncratic factor in the margins. Finally, $sign\left(\rho_{jl}^{\mathbf{X}}\right) = sign\left(a_j a_l\right)$ and therefore both positive and negative correlation can be accommodated.

Further, the pairwise linear correlation between the margin processes can be expressed in terms of the correlation between each margin and the systematic component as

$$\mathbb{C}orr\left(X_j(t), Z(t)\right) = a_j \sqrt{\frac{\mathbb{V}ar\left(Z(1)\right)}{\mathbb{V}ar\left(X_j(1)\right)}} \qquad \forall j = 1, ..., n, \tag{2.4}$$

implying that $\rho_{jl}^{\mathbf{X}} = \mathbb{C}orr\left(X_j(t), Z(t)\right)\mathbb{C}orr\left(X_l(t), Z(t)\right)$.

The multidimensional modelling approach put forward in this section is quite flexible as it allows to specify any univariate process for each of the components; however, the linear combination $\mathbf{Y}(t) + \mathbf{a}Z(t)$ returns a known distribution for the process $\mathbf{X}(t)$ only if suitable convolution conditions on the processes X, Y and Z are imposed, so that for $j = 1, ..., n$

$$X_j(t) \stackrel{d}{=} Y_j(t) + a_j Z(t) \tag{2.5}$$

(i.e. the equality is intended in distribution). This could be particularly convenient in the case in which the multivariate Lévy process $\mathbf{X}(t)$ is used to build a model for financial assets which is consistent with the information provided by traded derivative contracts. In this way, in fact, the calibration of the marginal distribution to observable market data would be independent of

the fitting of the correlation matrix, and therefore the parameters governing the idiosyncratic and the systematic processes. These parameters would be recovered at a second stage through the correlation matrix and the relevant restrictions imposed by the convolution (2.5). In more details, to facilitate the convolution, we choose $\mathbf{X}(t)$, $\mathbf{Y}(t)$ and $Z(t)$ from the same family of processes and impose (2.5) by solving

$$\varphi_{X_j}(u) = \varphi_{Y_j}(u) + \varphi_Z(a_j u) \quad j = 1, 2, ..., n. \tag{2.6}$$

This implies that, if m is the number of parameters describing the processes $X_j(t)$, $Y_j(t)$ and $Z(t)$, and the parameters of the margin processes are given, for a known correlation matrix the fitting of the joint distribution requires $n(m+1)+m$ parameters. As shown by equation (2.3), we can recover the m parameters describing the common process $Z(t)$ and the n loadings a_j, $j = 1, 2, ..., n$, through the correlation matrix subject to relevant convolution conditions. The nm parameters of the idiosyncratic process $Y_j(t)$ would then be obtained by solving equation (2.6).

Examples illustrating the case of a multivariate subordinated Brownian motions and jump diffusion processes are discussed in the following sections.

2.2. Multivariate subordinated Brownian motions

A subordinated Brownian motion $X = (X(t) : t \geq 0)$ is a Lévy process obtained by observing a (arithmetic) Brownian motion on a time scale governed by an independent subordinator, i.e. an increasing, positive Lévy process. Hence $X(t)$ has general form

$$X(t) = \theta G(t) + \sigma W(G(t)), \qquad \theta \in \mathbb{R}, \sigma > 0, \tag{2.7}$$

where $W = (W(t) : t \geq 0)$ is a Brownian motion and $G = (G(t) : t \geq 0)$ is a subordinator independent of W. The resulting characteristic function is

$$\phi_X(u;t) = e^{t\varphi_G\left(iu\theta - u^2\frac{\sigma^2}{2}\right)}, \quad u \in \mathbb{R},$$

where $\varphi_G(\cdot)$ denotes the characteristic exponent of the subordinator.

In general, the parameters of the distribution of the subordinator are chosen so that $\mathbb{E}G(t) = t$, in order to guarantee that the stochastic clock $G(t)$ is an unbiased reflection of calendar time (see, for example, Madan et al., 1998). The law of the increments of $G(t)$ allows us to characterize the resulting process. There are different methods for choosing a subordinator which is suitable for financial modelling; one class of such processes which proves to be quite popular due to its mathematical tractability is the family of tempered stable subordinators, which have characteristic exponent

$$\varphi_G(u) = \frac{\alpha - 1}{\alpha k}\left[\left(1 - \frac{iuk}{1-\alpha}\right)^{\alpha} - 1\right], \quad u \in \mathbb{C}, \tag{2.8}$$

where $k > 0$ is the variance rate of $G(t)$ and $\alpha \in [0,1)$ is the index of stability. In particular, if $\alpha = 0$, expression (2.8) is to be understood in a limiting sense and $G(t)$ is a Gamma process so that $X(t)$ is a VG process (Madan and Seneta, 1990, Madan and Milne, 1991, Madan et al., 1998). If, instead, $\alpha = 1/2$, the subordinator follows an Inverse Gaussian process and $X(t)$ is the NIG process introduced by Barndorff-Nielsen (1995).

Constructing Lévy processes by subordination has particular economic appeal as, in first place, empirical evidence shows that stock log-returns are Gaussian but only under trade time, rather than standard calendar time (see, for example, Geman and Ané, 1996). Further, the time change construction recognizes that stock prices are largely driven by news, and the time between one piece of news and the next is random as is its impact. Finally, the representation of Lévy processes as subordinated Brownian motions offers a high degree of mathematical tractability as, once we operate under business time, log-returns are once again Gaussian and therefore the results derived for the Black-Scholes model still hold.

To build the multivariate version of a subordinated Brownian motion of the form (2.7), we follow Proposition 1 and let $Y_j(t)$ and $Z(t)$ be independent subordinated Brownian motions chosen from the same family of distributions. Then, $\mathbf{X}(t)$ is a multivariate subordinated Brownian motion with margins of the same distribution's class as $Y_j(t)$ and $Z(t)$ if the convolution condition (2.5) (or equivalently 2.6) is satisfied.

We note that in this construction dependence stems from both the subordinator and the associated Wiener process. In more details, if $Y_j(t)$ and $Z(t)$ are obtained by subordinating respectively a Brownian motion with drift $\beta_j \in \mathbb{R}$ and diffusion $\gamma_j > 0$ by an unbiased subordinator G_{Yj}, and a Brownian motion with drift $\beta_Z \in \mathbb{R}$ and diffusion $\gamma_Z > 0$ by an unbiased subordinator G_Z, equation (2.5) implies that

$$\begin{aligned} G_j(t) &= \frac{\beta_j}{\theta_j} G_{Yj}(t) + a_j \frac{\beta_Z}{\theta_j} G_Z(t) \\ &= \frac{\gamma_j^2}{\sigma_j^2} G_{Yj}(t) + a_j^2 \frac{\gamma_Z^2}{\sigma_j^2} G_Z(t), \end{aligned}$$

where both equalities are intended in distribution. This shows that, similarly to Luciano and Semeraro (2010), the subordinator of the margin process can be decomposed into a stochastic clock which is common to all names in a given basket, and a time change that is specific to each asset, capturing the idiosyncratic part of the overall trading activity. This split between a common and an idiosyncratic component is supported also by the empirical analysis performed by Lo and Wang (2000). As the subordinators are assumed to be unbiased, the above relation implies

$$\theta_j = \beta_j + a_j \beta_Z \tag{2.9}$$

$$\sigma_j^2 = \gamma_j^2 + a_j^2 \gamma_Z^2. \tag{2.10}$$

If G_j, G_{Yj}, for $j=1,...,n$ and G_Z are tempered stable subordinators with variance rate $k_j>0$, $\nu_j>0$ and $\nu_Z>0$ respectively, equations (2.6) and (2.8) further imply

$$\begin{cases} k_j\theta_j=\nu_Z a_j\beta_Z & j=1,...,n \\ k_j\sigma_j^2=\nu_Z a_j^2\gamma_Z^2 & j=1,...,n \end{cases} \tag{2.11}$$

and consequently $k_j=\nu_j\nu_Z/(\nu_j+\nu_Z)$. The subordinators, in fact, are assumed to have the same stability index in order to guarantee that $X_j(t)$, $Y_j(t)$ and $Z(t)$ belong to the same family of distributions.

Example 1 (The VG process). Let $G(t)$ be a gamma process, i.e. a tempered stable process with scale parameter $\alpha=0$; then $X(t)$ is a VG process with characteristic function

$$\phi_X(u;t)=\left(1-iu\theta k+u^2\frac{\sigma^2}{2}k\right)^{-\frac{t}{k}}, \qquad u\in\mathbb{R}.$$

Under the restrictions imposed by equation (2.11), $\mathbf{X}(t)$ is a multivariate VG process with margins' parameters (θ_j,σ_j,k_j) constructed as described above and characteristic function

$$\phi_{\mathbf{X}}(\mathbf{u};t)=\left(1-i\beta_Z\nu_Z\sum_{j=1}^n a_ju_j+\frac{\gamma_Z^2}{2}\nu_Z\left(\sum_{j=1}^n a_ju_j\right)^2\right)^{-\frac{t}{\nu_Z}}\prod_{j=1}^n\left(1-iu_j\beta_j\nu_j+u_j^2\frac{\gamma_j^2}{2}\nu_j\right)^{-\frac{t}{\nu_j}}.$$

The coefficient of pairwise correlation given by equation (2.3) in this case reads

$$\rho_{jl}^{\mathbf{X}}=\frac{a_ja_l\left(\gamma_Z^2+\beta_Z^2\nu_Z\right)}{\sqrt{\sigma_j^2+\theta_j^2k_j}\sqrt{\sigma_l^2+\theta_l^2k_l}}. \tag{2.12}$$

Example 2 (The NIG process). In the case in which the tempered stable subordinator $G(t)$ has scale parameter $\alpha=1/2$, i.e. is an Inverse Gaussian process, then $X(t)$ is a NIG process with characteristic function

$$\phi_X(u;t)=e^{\frac{t}{k}\left(1-\sqrt{1-2iu\theta k+u^2\sigma^2k}\right)}, \qquad u\in\mathbb{R}.$$

Under the convolution restrictions (2.11), the margins $X_j(t)$ are NIG processes with parameters (θ_j,σ_j,k_j) as constructed above. The resulting characteristic function of the multivariate NIG process is

$$\begin{aligned}\phi_{\mathbf{X}}(\mathbf{u};t) &= e^{t\varphi(\mathbf{u})}\\ \varphi(\mathbf{u}) &= \frac{1}{\nu_Z}\left(1-\sqrt{1-2i\beta_Z\nu_Z\sum_{j=1}^n a_ju_j+\gamma_Z^2\nu_Z\left(\sum_{j=1}^n a_ju_j\right)^2}\right)\\ &\quad+\sum_{j=1}^n\frac{1}{\nu_j}\left(1-\sqrt{1-2iu_j\beta_j\nu_j+u_j^2\gamma_j^2\nu_j}\right).\end{aligned}$$

Equation (2.12) describes the pairwise correlation coefficient also in this case.

As both the VG and NIG are 3-parameter processes, the number of parameters required for the joint fit, given the margins, is $(4n+3)$, of which $3+n$ are observed from the correlation matrix subject to conditions (2.11), and $3n$ are obtained solving (2.9)-(2.10).

2.3. Multivariate jump-diffusion (JD) process

An alternative representation of Lévy processes quite common in financial applications relies on the observation that stock prices appear to have small continuous movements most of the time (due, for example, to a temporary imbalance between demand and supply); sometimes though they experience large jumps upon the arrival of important information with more than just a marginal impact. By its very nature, important information arrives only at discrete points in time and the jumps it causes have finite activity. A motion portraying such a dynamic is a jump-diffusion process, which can be decomposed as the sum of a Brownian motion with drift and an independent compound Poisson process. Hence, a Lévy process in the JD class has form

$$X(t) = \mu t + \sigma W(t) + \sum_{k=1}^{N(t)} \xi(k), \quad \mu \in \mathbb{R}, \sigma > 0,$$

where $W = (W(t) : t \geq 0)$ is a Brownian motion, $N = (N(t) : t \geq 0)$ is a Poisson process counting the jumps of X and $\xi(k)$ are i.i.d. random variables capturing the jump sizes (severities). W, N and ξ are independent of each other.

We assume that the rate of arrival of the Poisson process is $\lambda > 0$. In this case, we say that the process $X(t)$ has parameters (μ, σ, λ) and jump sizes density f_ξ; the resulting characteristic function is

$$\begin{aligned} \phi_X(u;t) &= e^{t\left(iu\mu - u^2\frac{\sigma^2}{2} + \lambda\left(\phi_\xi(u)-1\right)\right)}, \\ \phi_\xi(u) &= \mathbb{E}\left(e^{iu\xi}\right), \qquad u \in \mathbb{R}. \end{aligned}$$

Popular examples of JD processes used in finance are the so-called Merton process (Merton, 1976), for which the jump sizes are Gaussian, and the Kou process (Kou, 2002) in which case the jump sizes follow an asymmetric double exponential distribution.

In order to construct the multivariate version of the JD process, we follow the same steps as in the previous sections and let the idiosyncratic factor, Y_j, and the global factor, Z, to be two independent JD processes, respectively with parameters $(\beta_j, \gamma_j, \delta_j)$ and jump sizes' density f_{η_j}, and $(\beta_Z, \gamma_Z, \delta_Z)$ and jump sizes' density f_{η_Z}. Further, for the convolution property (2.5) to hold, i.e. for the process

$$\mathbf{X}(t) = \begin{pmatrix} X_1(t) \\ \vdots \\ X_n(t) \end{pmatrix} = \begin{pmatrix} Y_1(t) + a_1 Z(t) \\ \vdots \\ Y_n(t) + a_n Z(t) \end{pmatrix}$$

to be a multivariate JD process, whose margins have parameters $(\mu_j, \sigma_j, \lambda_j)$ and jump sizes' density f_{ξ_j}, we require $\mu_j = \beta_j + a_j\beta_Z$, $\sigma_j^2 = \gamma_j^2 + a_j^2\gamma_Z^2$ and

$$\lambda_j\left(\phi_{\xi_j}(u) - 1\right) = \delta_j\left(\phi_{\eta_j}(u) - 1\right) + \delta_Z\left(\phi_{\eta_Z}(a_j u) - 1\right), \quad u \in \mathbb{R}. \tag{2.13}$$

The corresponding pairwise correlation coefficient is

$$\rho_{jl}^{\mathbf{X}} = \frac{a_j a_l \left(\gamma_Z^2 + \delta_Z \mathbb{E}\left(\eta_Z^2\right)\right)}{\sqrt{\sigma_j^2 + \lambda_j \mathbb{E}\left(\xi_j^2\right)}\sqrt{\sigma_l^2 + \lambda_l \mathbb{E}\left(\xi_l^2\right)}}. \tag{2.14}$$

Thus, we note that under the proposed construction, the compound Poisson process components are allowed to jump at different points in time. Further, the convolution conditions reported above show the decomposition of both the continuous part of the risk and the pure jump one into their corresponding asset specific part and the one common to the entire basket under consideration.

As the Poisson process is closed under convolution, we could further assume $\lambda_j = \delta_j + \delta_Z$ for $j = 1, 2, ..., n$, so that equation (2.13) implies the following convolution on the distribution of the jump sizes:

$$\phi_{\xi_j}(u) = \frac{\delta_j \phi_{\eta_j}(u) + \delta_Z \phi_{\eta_Z}(a_j u)}{\delta_j + \delta_Z}, \quad u \in \mathbb{R}. \tag{2.15}$$

The proposed construction of multivariate JD processes falls in the more general common Poisson shock framework, reviewed in Lindskog and McNeil (2003) and further extended by Brigo et al. (2007). In our case, we use only two different types of shock (systematic and idiosyncratic); however, the distribution of the jump sizes depends on the nature of the underlying shock.

We note that a trivial solution to equation (2.15) can be obtained by assuming that ξ_j, η_j and $a_j \eta_Z$ are identically distributed. This is the case discussed by Moosbrucker (2006.b). However, in the following we do not consider this alternative as it imposes the unrealistic restriction that the jump sizes of each margin and the ones of its idiosyncratic component are identically distributed. Therefore, we make use of the (numerical) solution of equation (2.15).

Example 3 (The Merton process). Assume that the distribution of the jump sizes is Gaussian. Then, if $\eta_j \sim N\left(\vartheta_{Yj}, \upsilon_{Yj}^2\right)$ and $\eta_Z \sim N\left(\vartheta_Z, \upsilon_Z^2\right)$, the process $X_j(t) = Y_j(t) + a_j Z(t)$ is a Merton JD process with parameters $(\mu_j, \sigma_j, \lambda_j)$ as defined above, and jump sizes $\xi_j \sim N\left(\vartheta_j, \upsilon_j^2\right)$, where ϑ_j and υ_j are the solutions of

$$e^{iu\vartheta_j - u^2 \frac{\upsilon_j^2}{2}} = \frac{\delta_j e^{iu\vartheta_{Yj} - u^2 \frac{\upsilon_{Yj}^2}{2}} + \delta_Z e^{iua_j\vartheta_Z - u^2 \frac{a_j^2 \upsilon_Z^2}{2}}}{\delta_j + \delta_Z}, \qquad u \in \mathbb{R}. \tag{2.16}$$

The above implies

$$\begin{aligned} \vartheta_j &= \frac{\delta_j \vartheta_{Yj} + \delta_Z a_j \vartheta_Z}{\delta_j + \delta_Z}, \\ \upsilon_j^2 &= \frac{\delta_j(\vartheta_{Yj}^2 + \upsilon_{Yj}^2) + \delta_Z a_j^2 (\vartheta_Z^2 + \upsilon_Z^2)}{\delta_j + \delta_Z} - \vartheta_j^2. \end{aligned}$$

The coefficient of correlation is given by equation (2.14), with

$$\begin{aligned} \mathbb{E}\left(\xi_j^2\right) &= \vartheta_j^2 + \upsilon_j^2, \qquad \forall j = 1, ..., n \\ \mathbb{E}\left(\eta_Z^2\right) &= \vartheta_Z^2 + \upsilon_Z^2. \end{aligned}$$

Example 4 (The Kou process). In the case of the Kou process, the jump sizes follow a double exponential distribution with parameters (p, α^+, α^-), i.e. their density function is given by

$$p\alpha^+ e^{-\alpha^+ y} 1_{(y\geq 0)} + (1-p)\,\alpha^- e^{\alpha^- y} 1_{(y<0)}, \quad \alpha^+, \alpha^- \in \mathbb{R}^{++}, p \in [0,1].$$

Thus, if η_j, η_Z, ξ_j have a double exponential distribution respectively with parameters $(p_{Yj}, \alpha^+_{Yj}, \alpha^-_{Yj})$, $(p_Z, \alpha^+_Z, \alpha^-_Z)$, $(p_j, \alpha^+_j, \alpha^-_j)$, then, for the convolution condition (2.15) to hold, these parameters must satisfy the following

$$\begin{aligned} &p_j \frac{\alpha^+_j}{\alpha^+_j - iu} + (1-p_j)\frac{\alpha^-_j}{\alpha^-_j + iu} = \\ &\frac{1}{\delta_j + \delta_Z}\left[p_{Yj}\frac{\delta_j \alpha^+_{Yj}}{\alpha^+_{Yj} - iu} + (1-p_{Yj})\frac{\delta_j \alpha^-_{Yj}}{\alpha^-_{Yj} + iu} + p_Z \frac{\delta_Z \alpha^+_Z}{\alpha^+_Z - ia_j u} + (1-p_Z)\frac{\delta_Z \alpha^-_Z}{\alpha^-_Z + ia_j u}\right]. \end{aligned} \tag{2.17}$$

The correlation coefficient is obtained from equation (2.14) for

$$\begin{aligned} \mathbb{E}\left(\xi_j^2\right) &= 2\left(\frac{p_j}{\left(\alpha^+_j\right)^2} + \frac{1-p_j}{\left(\alpha^-_j\right)^2}\right), \qquad \forall j = 1, ..., n \\ \mathbb{E}\left(\eta_Z^2\right) &= 2\left(\frac{p_Z}{\left(\alpha^+_Z\right)^2} + \frac{1-p_Z}{\left(\alpha^-_Z\right)^2}\right). \end{aligned}$$

Finally, we note that for the multivariate Kou model, the reconstruction of the margin parameters (p, α^+, α^-) from the components parameters can only be performed numerically.

In the following, we consider applications of our multivariate approach to option pricing problems; therefore, without loss of generality, we consider the case of a JD process with no drift, i.e. we set $\mu_j = \beta_j = \beta_Z = 0$ for $j = 1, ..., n$. This implies that for the joint fit, given the margins, we require $(5n+4)$ parameters in the case of the Merton process and $(6n+5)$ in the case of the Kou process.

3. Multivariate asset modelling: calibration and derivative pricing

In this section we discuss how the multivariate Lévy process constructed in the previous part of this paper can be used to set up a model of the financial market for option pricing. In particular, we analyze the calibration of the model to market data in view of applications to the problem of pricing multi-assets products.

To this purpose, we consider a frictionless market in which asset log-returns are modelled by the multivariate Lévy process defined in Proposition 1, so that under any risk neutral martingale measure $\mathbb{P}$ asset prices are given by

$$S_j(t) = S_j(0) e^{\left(r - q_j - \varpi_j\right)t + X_j(t)}, \quad j = 1, ..., n$$

where $r > 0$ is the risk free rate of interest, $S_j(0)$ and q_j denote respectively the spot price and the dividend yield of the jth asset, $X_j(t)$ is the jth component of the multivariate Lévy process, and ϖ_j denotes the exponential compensator of $X_j(t)$.

Table 1 Synopsis of market data for Ford Motor Company, Abbott Laboratories and Baxter International Inc.

VALUATION DATE	ASSET	$S(0)$	q	125-day correlation F (Ford)	ABT (Abbott Lab.)	BAX (Baxter)
30/09/2008	F	5.20	0.0%	100%		
	ABT	57.58	2.8%	25%	100%	
	BAX	65.67	1.5%	30%	64%	100%
27/02/2009	F	2.00	0.0%	100%		
	ABT	47.34	3.0%	37%	100%	
	BAX	50.91	1.8%	34%	83%	100%
30/09/2009	F	7.21	0.0%	100%		
	ABT	49.47	3.0%	-22%	100%	
	BAX	57.02	1.7%	-15%	45%	100%

Note. Correlation matrix estimated using historical log-returns of the three assets over a 125-day time window, up to (and including) the valuation date. Source: Bloomberg.

We note that a full calibration procedure aimed at extracting the correlation coefficient as well should also make use of derivatives written on all the assets, like for example a basket option. However, as these products are only traded on the OTC market, the implied correlation cannot be estimated at this stage. For this reason, in this application we use as a proxy the historical pairwise correlation coefficient $\rho^{\mathbf{X}}_{jl}$ between asset log-returns. In the final part of this section, we illustrate how information about the correlation between assets could be potentially recovered from market prices, avoiding in this way historical estimation.

3.1. Market data

We test the flexibility of the model by calibrating it to option prices on Ford Motor Company, Abbott Laboratories and Baxter International Inc. We use Bloomberg quotes at three different valuation dates, September 30, 2008, February 27, 2009 and September 30, 2009, in order to explore the behaviour of the proposed model when fitting different correlation values. A synopsis of the three assets is reported in Table 1. The risk free rate of interest is taken from Bloomberg as well in correspondence of the relevant dates. Historical correlation between assets log-returns has been estimated on a time window of 125 days up to (and including) the valuation date.

The three assets considered in this analysis are constituents of the S&P100 index, and represent three different industries: automotive, drug manufacturers and medical instruments and supplies respectively. Abbott Laboratories and Baxter International Inc. are part of the same healthcare sector. Further, from Table 1 we observe that in September 2008 the three assets exhibit positive correlation, at a level which is fairly similar between Ford and the remaining two assets, whilst it is significantly higher between Abbott and Baxter. This date, in fact, coincides with the peak of the financial crisis which led to the collapse of Lehman Brothers; the car industry was also experiencing a particularly difficult period following the General Motors liquidity crisis and the sales fall also

reported by its main competitors. Correlation values further increase in February 2009, when the effects of the credit crisis are fully captured by the estimation procedure used in this analysis. These observations lead us to expect the common component $Z(t)$ to play a significant role in the prices of Abbott Lab. and Baxter, whilst we expect it to have a smaller impact on Ford prices. The same consideration holds especially for the September 2009 valuation date, when Ford exhibits negative correlation with the other two assets considered in this analysis.

3.2. Model calibration

The calibration of the proposed multivariate asset model is performed in steps. In first place, we extract the parameters of the margin processes, $\mathbf{X}(t)$, using the bid and ask implied volatilities of European call options written on each asset; in more details, we minimize a weighted root mean squared error, where the error is defined as

$$\begin{cases} \bar{C}_{bid} - C_m & \text{if } C_m < \bar{C}_{bid} \\ 0 & \text{if } \bar{C}_{bid} < C_m < \bar{C}_{ask} \\ C_m - \bar{C}_{ask} & \text{if } C_m > \bar{C}_{ask}. \end{cases}$$

$\bar{C}$ denotes the market quote; the weights are obtained using a procedure based on Huang and Wu (2004). Model prices, denoted by C_m, are computed using the Fourier inversion procedures of Carr and Madan (1999); out-of-the-money options are dealt with the time value approach.

The second step consists in the calibration of the parameters of the idiosyncratic process, $\mathbf{Y}(t)$, and the systematic component, $Z(t)$, by fitting the correlation matrix using least squares, and imposing the relevant convolution conditions.

The calibrated parameters of the margins, the idiosyncratic components and the systematic process are reported in Tables 2-5 for all the valuation dates considered and models analysed in this paper. We note that the convolution conditions can only be solved numerically which, potentially, could introduce an approximation error. In the attempt of quantifying it, in the Tables we also report the difference between the moments of the distribution of the processes $\mathbf{X}(t)$, calculated using the components parameters in conjunction with equation (2.2), and the same moments calculated instead using the margin parameters and the exact formulae reported in Appendix B (equations B.7-B.10) and Appendix C (equations C.1-C.4). Further, Figure 1 show the QQ plots of the (simulated) samples of the margin process obtained by direct calibration to European vanilla options and the same process obtained, instead, by linear combination of the idiosyncratic process and the systematic process. In particular, in these plots we consider the case of the multivariate VG model (similar results have been obtained for the other models presented in this paper and are available from the authors). These results illustrate the goodness of the convolution provided by the fitting procedure, although the accuracy of the approximation tends to deteriorate at the very

Table 2 **Calibration of the multivariate VG model.**

VG MODEL									
	VALUATION DATE								
	30/09/2008			27/02/2009			30/09/2009		
MARGINS	F	ABT	BAX	F	ABT	BAX	F	ABT	BAX
θ	-2.6871	-0.6373	-0.5286	-6.3009	-0.8664	-0.7969	0.4058	-0.2283	-0.5425
σ	0.8537	0.2259	0.2296	0.5354	0.1509	0.2613	0.6040	0.2352	0.2129
k	0.0264	0.0928	0.0897	0.0588	0.1555	0.0805	0.0104	0.2339	0.0944
RMSE	3.75E-02	1.23E-01	1.49E-01	4.54E-02	2.72E-01	3.60E-01	4.78E-02	7.15E-02	1.01E-01
(w)RMSE	2.21E-03	9.94E-03	1.18E-02	4.64E-03	1.51E-02	1.61E-02	1.86E-03	4.09E-03	3.89E-03
IDIOSYNCRATIC PART									
β	-2.1117	-0.2552	-0.1467	-4.9115	-0.0838	-0.1316	0.2888	-0.1168	-0.4356
γ	0.8120	0.1429	0.1488	0.4710	0.0469	0.2311	0.5788	0.1682	0.1431
ν	0.0318	0.2316	0.2137	0.0892	1.6068	0.1512	0.0106	0.4570	0.1176
a	1.1564	0.7678	0.7675	1.4550	0.8197	0.6969	-0.9348	0.8903	0.8541
SYSTEMIC PART									
β_Z	-0.4976			-0.9547			-0.1252		
γ_Z	0.2278			0.1750			0.1846		
ν_Z	0.1547			0.1721			0.4790		
MOMENT MATCHING ERROR									
$\mathbb{E}X(1)$	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
$stdX(1)$	-1.34E-03	0.00E+00	-3.72E-03	-4.62E-02	-1.08E-14	-3.80E-02	-4.72E-03	-2.36E-14	-1.33E-14
$\gamma_1(1)$	-3.61E-03	2.11E-15	7.77E-03	3.41E-02	6.64E-14	-1.91E-02	-1.80E-02	1.11E-12	-1.58E-07
$\gamma_2(1)$	-5.32E-05	-5.27E-15	-2.25E-02	-6.16E-02	-2.08E-13	-5.84E-02	-1.28E-02	-4.05E-12	-8.48E-02
CALIBRATION ERROR									
RMSE	9.49E-03	1.70E-07	1.18E-07	9.71E-09	-5.14E-08	-1.30E-08	3.69E-09	0.00E+00	1.11E-09
(w)RMSE	8.81E-04	-1.17E-08	2.45E-09	1.17E-09	-3.61E-10	-7.81E-10	9.43E-11	0.00E+00	5.45E-11
CORRELATION ERROR									
F	-			-			-		
ABT	3.05E-02	-		5.43E-08	-		4.60E-07	-	
BAX	-5.84E-04	5.47E-08	-	1.39E-07	1.22E-07	-	-5.28E-02	-9.41E-07	-

Note. Parameters of the margins, the systemic part and the idiosyncratic components as at 30/09/2009, 27/02/2009 and 30/09/2009. Parameters of the marginal distributions $(\theta_j, \sigma_j, k_j)$ obtained by direct calibration to market prices. Parameters governing the idiosyncratic risk process, $(\beta_j, \gamma_j, \nu_j, a_j)$, and the systematic risk process, $(\beta_Z, \gamma_Z, \nu_Z)$, obtained by fitting the correlation matrix and then solving the parameters conditions given in Example 1. Moment matching error: the difference between the exact moments provided in Appendix B (calculated using the parameters of the marginal process) and the moments reconstructed using equation (2.2). Calibration error: difference between the errors produced by the calibration to market option prices of the margin processes, $\mathbf{X}(t)$, and the linear transformation $\mathbf{Y}(t) + \mathbf{a}Z(t)$. Correlation error: difference between the model and the sample correlation.

far end of the tails. Tables 2-5 also report the error produced by the multivariate construction in fitting the given correlation matrices. In particular, the full range of correlation values provided by the data is captured with a satisfactory degree of accuracy.

As a further test, we re-calculate the prices of the European vanilla options using the joint characteristic function and quantify the error against the corresponding market data, as reported in Tables 2-5. The (weighted) root mean squared errors are very close to the ones generated by

Table 3 **Calibration of the multivariate NIG model.**

NIG MODEL									
	VALUATION DATE								
	30/09/2008			27/02/2009			30/09/2009		
MARGINS	F	ABT	BAX	F	ABT	BAX	F	ABT	BAX
θ	-2.0985	-0.3917	-0.3879	-6.2583	-0.8635	-0.8041	0.5358	-0.2567	-0.5414
σ	0.8082	0.2206	0.2141	0.9382	0.2350	0.2570	0.5968	0.2303	0.2167
k	0.0175	0.0698	0.0559	0.0397	0.1140	0.0881	0.0196	0.2536	0.0937
RMSE	3.63E-02	1.20E-01	1.40E-01	4.25E-02	2.72E-01	3.60E-01	5.00E-02	7.15E-02	9.79E-02
(w)RMSE	2.11E-03	9.70E-03	1.13E-02	4.44E-03	1.50E-02	1.60E-02	1.94E-03	4.09E-03	3.76E-03
IDIOSYNCRATIC PART									
β	-1.7346	-0.1828	-0.1483	-4.9265	-0.0836	-0.1328	0.4072	-0.0783	-0.4024
γ	0.7579	0.1507	0.1081	0.8572	0.0731	0.1706	0.5806	0.1271	0.1569
ν	0.0201	0.1495	0.0976	0.0580	1.1777	0.2917	0.0207	0.8316	0.1260
a	1.1480	0.6591	0.7559	1.3965	0.8178	0.7039	-0.6866	0.9523	0.7420
SYSTEMIC PART									
β_Z	-0.3170			-0.9537			-0.1874		
γ_Z	0.2445			0.2731			0.2016		
ν_Z	0.1308			0.1262			0.3648		
MOMENT MATCHING ERROR									
$\mathbb{E}X(1)$	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
$stdX(1)$	-6.22E-04	0.00E+00	-2.66E-03	-2.39E-02	-6.48E-13	-7.14E-03	-3.18E-03	-3.35E-13	-8.07E-14
$\gamma_1(1)$	-1.81E-04	4.44E-16	6.64E-02	3.02E-02	6.29E-12	1.23E-03	-2.61E-03	1.54E-11	-2.24E-07
$\gamma_2(1)$	-2.40E-04	-1.39E-15	-1.39E-01	-6.35E-02	-2.36E-11	-4.23E-02	-3.76E-03	-7.30E-11	-2.83E-02
CALIBRATION ERROR									
RMSE	3.82E-02	2.86E-01	2.95E-01	-1.85E-09	4.90E-09	-3.91E-09	4.16E-09	0.00E+00	-4.86E-11
(w)RMSE	3.00E-03	2.27E-02	3.04E-02	-2.35E-10	1.19E-11	9.51E-10	9.68E-11	0.00E+00	3.12E-10
CORRELATION ERROR									
F	-			-			-		
ABT	1.50E-02	-		6.43E-07	-		9.88E-07	-	
BAX	1.80E-02	-5.48E-08	-	1.80E-07	4.38E-07	-	-1.59E-02	7.42E-02	-

Note. Parameters of the margins, the systemic part and the idiosyncratic components as at 30/09/2009, 27/02/2009 and 30/09/2009. Parameters of the marginal distributions $(\theta_j, \sigma_j, k_j)$ obtained by direct calibration to market prices. Parameters governing the idiosyncratic risk process, $(\beta_j, \gamma_j, \nu_j, a_j)$, and the systematic risk process, $(\beta_Z, \gamma_Z, \nu_Z)$, obtained by fitting the correlation matrix and then solving the parameters conditions given in Example 2. Moment matching error: difference between the exact moments provided in Appendix B (calculated using the parameters of the marginal process) and the moments reconstructed using equation (2.2). Calibration error: difference between the errors produced by the calibration to market option prices of the margin processes, $\mathbf{X}(t)$, and the linear transformation $\mathbf{Y}(t) + \mathbf{a}Z(t)$. Correlation error: difference between the model and the sample correlation.

direct calibration of the marginal distribution, which shows that any potential approximation error introduced by the joint fitting procedure is relatively negligible for this type of application. We note though that the higher the number of parameters in the joint distribution, the less flexible the fitting of the multivariate model, and an initial guess which is quite close to the optimal solution is required for the fitting procedure to converge efficiently. Finally, Figure 2 shows the volatilities recovered by the standard Black-Scholes formula, in the case in which the input prices are generated

Table 4 **Calibration of the multivariate Merton jump diffusion model.**

MERTON MODEL									
	VALUATION DATE								
	30/09/2008			27/02/2009			30/09/2009		
MARGINS	F	ABT	BAX	F	ABT	BAX	F	ABT	BAX
σ	0.8232	0.2553	0.2353	0.9462	0.2193	0.2551	0.5858	0.2085	0.2219
λ	0.6969	0.1974	0.2017	2.3781	0.5619	0.5674	0.2708	0.2055	0.2186
ϑ	-0.4738	-0.2600	-0.2837	-0.6313	-0.3574	-0.2644	0.0177	-0.2099	-0.3031
υ	0.2770	0.1998	0.1867	0.4650	0.2251	0.1800	0.3120	0.2772	0.1500
RMSE	5.03E-02	1.53E-01	1.46E-01	3.69E-02	2.74E-01	3.60E-01	4.67E-02	6.97E-02	9.79E-02
(w)RMSE	3.37E-03	1.23E-02	1.20E-02	3.87E-03	1.51E-02	1.60E-02	1.82E-03	3.10E-03	3.88E-03
IDIOSYNCRATIC PART									
γ	0.7748	0.1751	0.1018	0.8909	0.1000	0.1960	0.5682	0.1000	0.1791
δ	0.5183	0.0187	0.0231	1.9165	0.1003	0.1057	0.1683	0.1030	0.1161
ϑ_Y	-0.5267	-0.5807	-0.3502	-0.6257	0.1445	0.0491	-0.0578	-0.2320	-0.4288
υ_Y	0.2736	0.1252	0.2651	0.4881	0.1469	0.0100	0.3352	0.2792	0.0784
a	0.6467	0.4320	0.4936	0.8863	0.5428	0.4544	-0.5606	0.7188	0.5148
SYSTEMIC PART									
γ_Z	0.4299			0.3595			0.2546		
δ_Z	0.1787			0.4617			0.1025		
ϑ_Z	-0.5073			-0.7437			-0.2695		
υ_Z	0.3062			0.3900			0.3738		
MOMENT MATCHING ERROR									
$\mathbb{E}X(1)$	1.40E-03	-1.30E-03	-4.42E-03	2.09E-03	-3.37E-06	8.19E-04	-9.76E-04	6.04E-04	-2.27E-03
$stdX(1)$	5.89E-04	4.99E-03	6.38E-03	-1.74E-04	1.26E-04	-1.32E-02	1.09E-04	6.30E-05	-5.12E-03
$\gamma_1(1)$	4.16E-04	-4.23E-02	–6.42E-02	-1.41E-03	9.06E-03	9.32E-02	1.03E-02	4.41E-03	3.51E-02
$\gamma_2(1)$	1.83E-03	7.07E-02	9.96E-02	-2.24E-04	-1.98E-02	-1.76E-01	-2.02E-03	5.89E-03	-5.94E-02
CALIBRATION ERROR									
RMSE	3.92E-05	6.05E-03	6.69E-04	-9.62E-06	-7.44E-05	2.63E-02	5.80E-05	-6.14E-04	-1.11E-03
(w)RMSE	3.92E-06	5.20E-04	5.97E-04	-1.80E-06	-4.99E-07	8.55E-04	2.05E-06	-3.40E-05	-8.55E-05
CORRELATION ERROR									
F	-			-			-		
ABT	-2.08E-04	-		-3.87E-07	-		5.03E-04	-	
BAX	-9.11E-04	3.56E-04	-	-3.56E-07	-8.68E-07	-	-7.91E-04	-3.47E-04	-

Note. Parameters of the margins, the systemic part and the idiosyncratic components as at 30/09/2009, 27/02/2009 and 30/09/2009. Parameters of the marginal distributions $(\sigma_j, \lambda_j, \vartheta_j, \upsilon_j)$ obtained by direct calibration to market prices. Parameters governing the idiosyncratic risk process, $(\gamma_j, \delta_j, \vartheta_{Yj}, \upsilon_{Yj}, a_j)$, and the systematic risk process, $(\gamma_Z, \delta_Z, \vartheta_Z, \upsilon_Z)$, obtained by fitting the correlation matrix and then solving the parameters conditions given in Example 3. Moment matching error: difference between the exact moments provided in Appendix C (calculated using the parameters of the marginal process) and the moments reconstructed using equation (2.2). Calibration error: difference between the errors produced by the calibration to market option prices of the margin processes, $\mathbf{X}(t)$, and the linear transformation $\mathbf{Y}(t) + \mathbf{a}Z(t)$. Correlation error: difference between the model and the sample correlation.

by the multidimensional VG process. The plot also reports the original bid-ask volatilities obtained from market data for vanilla options.

We conclude by noting that, as in the set up proposed in this paper the correlation coefficient is an explicit function of the model parameters, market consistent information on the (in general not

Table 5 **Calibration of the multivariate Kou jump diffusion model.**

KOU MODEL									
	VALUATION DATE								
	30/09/2008			27/02/2009			30/09/2009		
MARGINS	F	ABT	BAX	F	ABT	BAX	F	ABT	BAX
σ	0.8116	0.2549	0.2500	1.1776	0.2234	0.2794	0.5828	0.2300	0.2432
λ	0.2549	0.2000	0.2170	0.9873	0.9939	0.5105	0.3624	0.2033	0.2115
p	0.0254	0.3000	0.6184	0.0579	0.0772	0.0658	0.2385	0.4600	0.2300
α^+	21.3413	22.1345	26.9924	5.5704	19.7977	11.7134	3.5667	8.0000	22.4224
α^-	2.4050	3.5001	2.7864	2.0662	4.1458	4.5614	7.3701	4.3438	4.8337
RMSE	5.11E-02	1.63E-01	1.55E-01	3.77E-02	2.71E-01	3.56E-01	4.81E-02	1.25E-01	1.25E-01
(w)RMSE	3.57E-03	1.31E-02	1.29E-02	4.51E-03	1.49E-02	1.59E-02	1.87E-03	6.66E-03	5.17E-03
IDIOSYNCRATIC PART									
γ	0.7667	0.1639	0.1169	1.1446	0.1042	0.2278	0.5629	0.1216	0.2000
δ	0.0668	0.0119	0.0289	0.4967	0.5033	0.0198	0.2424	0.0833	0.0915
p_Y	0.0115	0.0568	0.4937	0.0100	0.9000	0.1000	0.2681	0.2185	0.2031
α_Y^+	6.3406	6.4452	41.0497	1.9113	4.7301	38.6608	3.4307	4.5397	8.8803
α_Y^-	1.7582	1.9443	1.8969	2.0154	5.0739	87.2172	6.3361	4.5398	4.1503
a	0.5455	0.4003	0.4530	0.9538	0.6813	0.5581	-0.5226	0.6754	0.4788
SYSTEMIC PART									
γ_Z	0.4878			0.2900			0.2891		
δ_Z	0.1881			0.4906			0.1200		
p_Z	0.1908			0.0204			0.3709		
α_Z^+	2.3883			6.7138			22.9985		
α_Z^-	2.3863			2.0884			3.2841		
MOMENT MATCHING ERROR									
$\mathbb{E}X(1)$	-3.90E-02	-1.21E-02	4.70E-03	1.96E-02	-1.47E-01	2.61E-02	-1.52E-02	1.10E-02	-5.97E-03
$stdX(1)$	1.37E-02	1.13E-02	-1.41E-06	-2.34E-03	-4.74E-02	-3.05E-02	0.00E+00	2.55E-05	3.62E-03
$\gamma_1(1)$	-3.41E-02	-2.14E-01	-2.16E-01	-1.32E-03	-3.62E-01	2.93E-01	0.00E+00	1.20E-04	-4.84E-02
$\gamma_2(1)$	-2.13E-02	-4.83E-01	1.12E-04	2.53E-03	-9.59E-01	-1.02E+00	0.00E+00	-6.71E-04	-3.02E-04
CALIBRATION ERROR									
RMSE	2.56E-02	5.37E-02	6.71E-02	-6.49E-04	4.68E-01	6.07E-02	-1.35E-12	-5.20E-05	8.08E-03
(w)RMSE	1.72E-03	4.32E-03	5.75E-03	-7.79E-05	2.35E-02	2.07E-03	2.12E-12	-3.44E-06	3.83E-04
CORRELATION ERROR									
F	-			-			-		
ABT	9.16E-03	-		1.06E-04	-		-2.57E-04	-	
BAX	0.00E+00	2.89E-08	-	6.25E-04	-2.44E-04	-	1.90E-04	-5.49E-04	-

Note. Parameters of the margins, the systemic part and the idiosyncratic components as at 30/09/2009, 27/02/2009 and 30/09/2009. Parameters of the marginal distributions $(\sigma_j, \lambda_j, p_j, \alpha_j^+, \alpha_j^-)$ obtained by direct calibration to market prices. Parameters governing the idiosyncratic risk process, $(\gamma_j, \delta_j, p_{Yj}, \alpha_{Yj}^+, \alpha_{Yj}^-, a_j)$, and the systematic risk process, $(\gamma_Z, \delta_Z, p_Z, \alpha_Z^+, \alpha_Z^-)$, obtained by fitting the correlation matrix and then solving the parameters conditions given in Example 4. Moment matching error: difference between the exact moments provided in Appendix C (calculated using the parameters of the marginal process) and the moments reconstructed using equation (2.2). Calibration error: difference between the errors produced by the calibration to market option prices of the margin processes, $\mathbf{X}(t)$, and the linear transformation $\mathbf{Y}(t) + \mathbf{a}Z(t)$. Correlation error: difference between the model and the sample correlation.

Table 6 **Correlation between asset log-returns (X) and the common component (Z) and the idiosyncratic part (Y).**

	VALUATION DATE											
	30/09/2008				27/02/2009				30/09/2009			
	VG MODEL		NIG MODEL		VG MODEL		NIG MODEL		VG MODEL		NIG MODEL	
	Z	**Y**	Z	**Y**	Z	**Y**	Z	**Y**	Z	**Y**	Z	**Y**
F	0.3622	0.9336	0.3628	0.9326	0.3893	0.9519	0.3893	0.9377	-0.3148	0.9573	-0.2639	0.9700
ABT	0.7713	0.6328	0.7303	0.6831	0.9504	0.3111	0.9504	0.3111	0.6987	0.7154	0.8337	0.5522
BAX	0.8265	0.5862	0.8764	0.5050	0.8733	0.6850	0.8733	0.5276	0.6440	0.7650	0.6287	0.7776
	JD Merton		JD Kou		JD Merton		JD Kou		JD Merton		JD Kou	
	Z	**Y**	Z	**Y**	Z	**Y**	Z	**Y**	Z	**Y**	Z	**Y**
F	0.3416	0.9392	0.3485	0.9203	0.3893	0.9212	0.3899	0.9228	-0.2713	0.9623	-0.2709	0.9626
ABT	0.7313	0.6569	0.7436	0.6105	0.9504	0.3100	0.9495	0.5932	0.8090	0.5874	0.8129	0.5822
BAX	0.8763	0.4339	0.8607	0.5091	0.8733	0.5604	0.8739	0.6458	0.5558	0.8538	0.5529	0.8171

Note. These values have been obtained using equation (2.4) and the parameters of the components.

observable) common component could be recovered directly from the market correlation matrix. The multivariate construction presented in section 2 would allow us to use this information to observe the impact of these components on each asset through, for example, the correlation coefficient (2.4). For the case of the assets considered in this study, these results are shown in Table 6. In particular, we observe the very strong impact of the systematic process on the correlation between the log-returns of Abbott Lab. and Baxter; the role of the common factor though is not so relevant in the case of Ford, confirming the economic considerations offered above. Further evidence is provided by the parameters reported in Tables 2-5; for example, in the case of the VG model specification, the systematic component explains only 13% of the total variance of Ford log-returns in September 2008, against the 60% of the total variance of Abbot Lab. and 67% of Baxter. This changes in February 2009 to 14% for Ford, 90% for Abbott Lab. and 62% for Baxter. In September 2009, the contribution of Z accounts for 10% of the total variation of Ford, and 49%-41% for Abbott Lab. and Baxter respectively. Similar considerations hold for the other models analyzed in this paper.

3.3. Pricing of Exotics and implied correlation

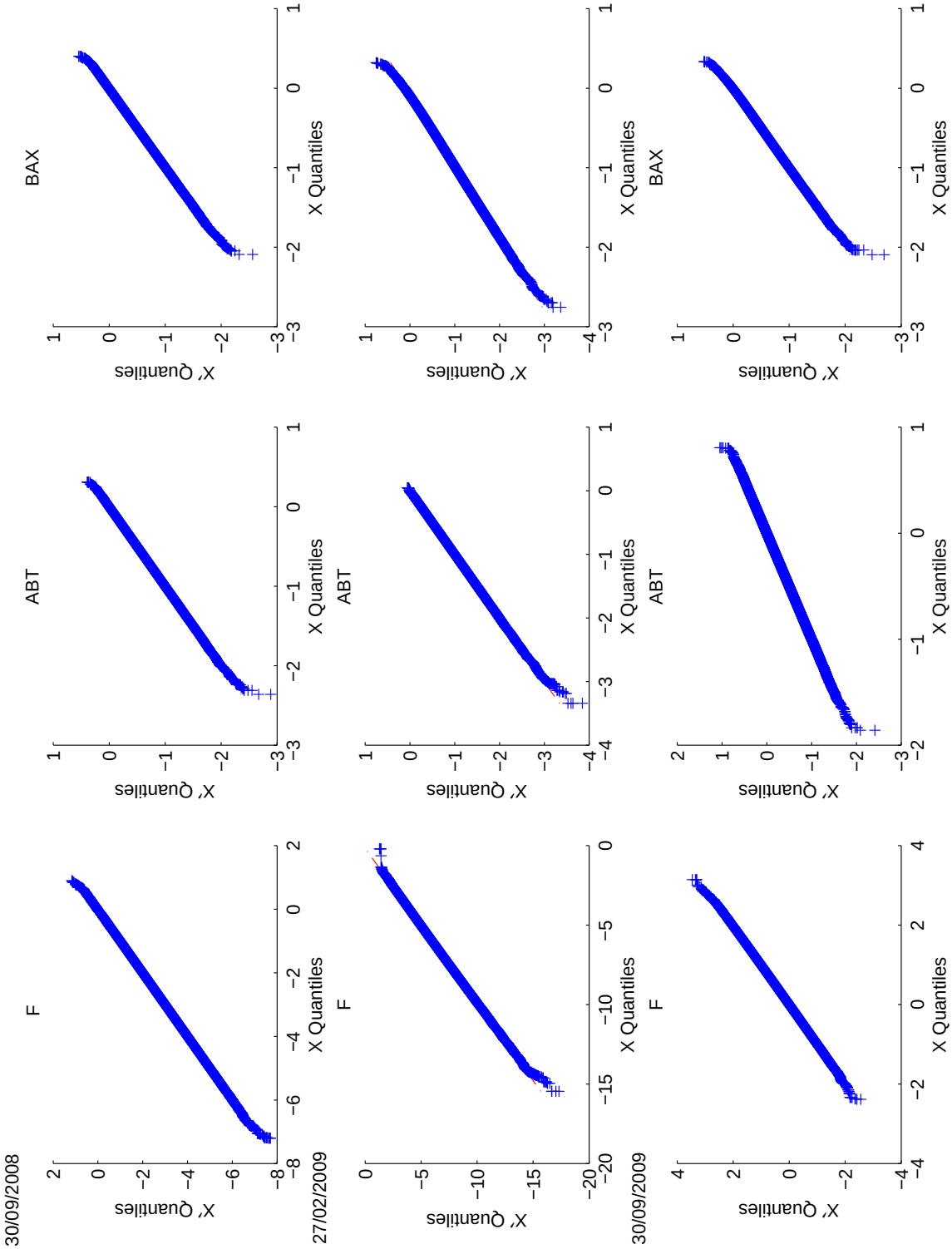
We conclude this section by considering the pricing of European style multinames products in the market model calibrated in section 3.2. In particular, we consider the case of a spread (call) option with payoff at maturity

$$(S_1(T) - S_2(T) - K)^+$$

where S_j, $j = 1, 2$ denote the prices of Baxter and Abbott respectively.

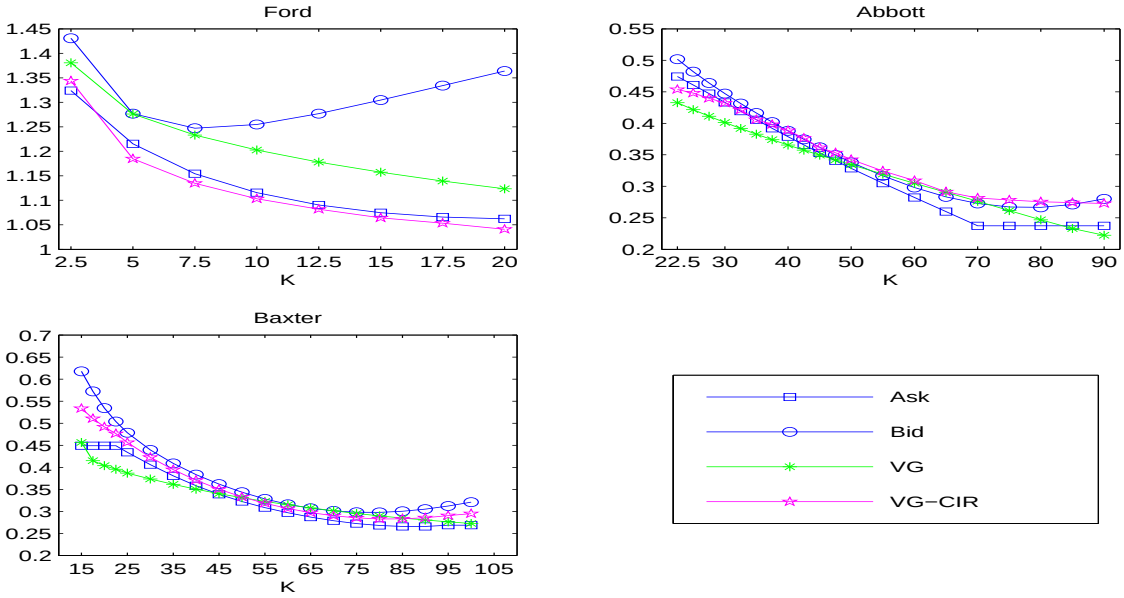
In this example, we assume a joint VG dynamics for the log-returns of the two assets, with parameters obtained by the joint model calibration reported in Table 2. Further, we assume that the valuation date is 27/02/2009. These prices are compared with the ones generated under the

Figure 1 **Convolution error: recovering the VG distribution.**



Note. QQ plots of a Monte Carlo sample of the margin VG process, $X(t)$, and the linear transformation process, $X'(t) = Y(t) + aZ(t)$, for Ford, Abbott Lab. and Baxter at September 30, 2008, February 27, 2009 and September 30, 2009. Monte Carlo simulation based on 1,000,000 iterations.

Figure 2 Recovering implied volatilities with multivariate VG processes and VG-CIR processes.

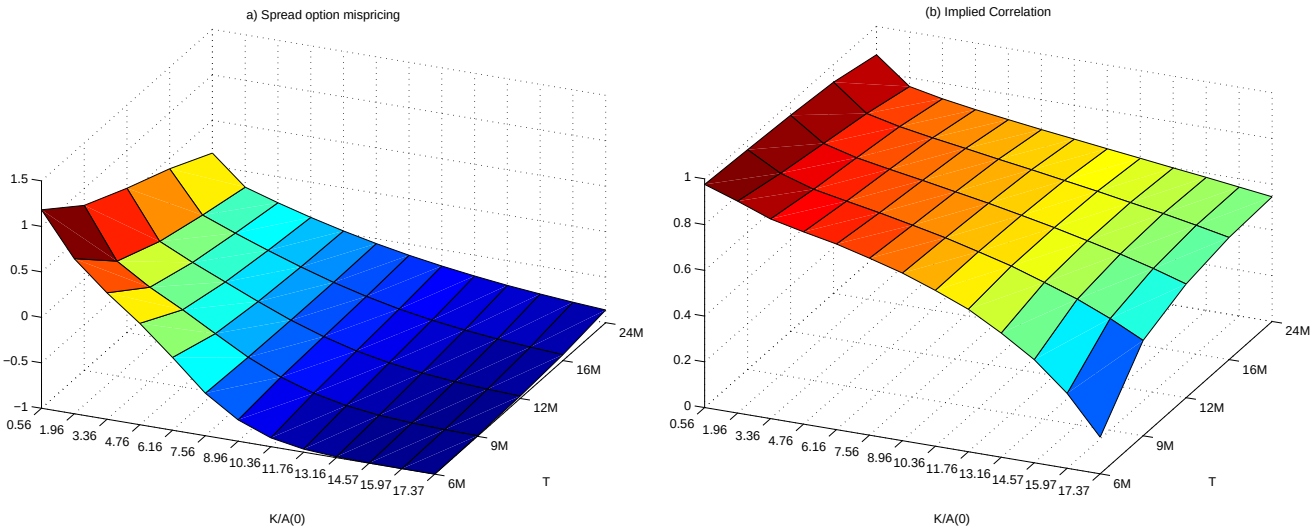


Note. Implied volatility curve recovered from the multivariate VG and VG-CIR models. Volatility obtained by inversion for the Black-Scholes formula in correspondence of input prices obtained using the given multivariate processes. Parameters: Tables 2 and 7. Maturity: 11 months; valuation date: 27/02/2009.

assumption of jointly log-normal asset prices (standard model) in order to extract information about the market consensus on asset correlations. All prices are computed using the Fourier inversion method proposed by Hurd and Zhou (2009). In more details, for the case of the standard model, we use as input the market implied volatility of each asset as extracted from vanilla option prices obtained under the VG model in correspondence of each strike and maturity considered in this example, and the historical correlation matrix presented in Table 1. The results are presented in Figure 3: in panel (a), we report the mispricing generated by the standard model with respect to the VG model; in panel (b), instead, we report the implied correlation extracted from the standard model using as input the spread option prices obtained under the multivariate VG model.

Figure 3.(a) shows that the standard model overprices the spread option for in-the-money contracts, whilst it underprices them for out-of-the-money contracts. The mispricing of in-the-money contracts is particularly evident for short maturities. This is due to the fact that the distribution of the spread underlying the contract has a higher variance and excess kurtosis under the standard model than under the VG model. These tend to reduce when the strike price increases (due to the trend of the implied volatilities of the components). In terms of implied correlation, this translates into higher values compared the historical correlation in the case of in-the-money options and lower values for the case of out-of-the-money and deep out-of-the-money options, as shown by Figure 3.(b). This observed "skew" pattern is consistent with the so-called correlation leverage

Figure 3 **Spread call options: multivariate VG model vs standard model. Mispricing and implied correlation.**



Note. Multivariate VG process parameters: Table 2. Standard model volatility parameters: implied volatility recovered from the vanilla options VG prices; correlation coefficient Table 1. Valuation date: 27/02/2009. $A(0) = S_1(0) - S_2(0)$. Prices obtained using the approach of Hurd and Zhou (2009). Mispricing: difference between standard model prices and VG prices, expressed as percentage of the VG prices. Implied correlation: recovered from the standard model with input prices set equal to the spread option prices obtained under the VG model.

effect observed for example by Da Fonseca et al. (2007). Implied correlation and historical one are close for at-the-money contracts (for which the mispricing is close to zero).

4. Extensions to multivariate time changed Lévy processes: a simple setting.

The multivariate Lévy process introduced in Proposition 1 can also be used as the basis for multivariate time changed Lévy processes constructions, allowing for the introduction of stochastic volatility features. Time changed Lévy processes represent a way to simultaneously and parsimoniously capture the fact that asset prices jump, return volatilities are stochastic and are correlated to asset returns. These processes have been studied in the context of option pricing by Carr et al. (2003), Carr and Wu, and Huang and Wu (2004) (see Appendix B).

Let $\mathbf{V}(t)$ be a n-dimensional time change; a multivariate time changed Lévy process $\mathbf{B}(t)$ can be then obtained by evaluating each component of a n-dimensional Lévy process $\mathbf{X}(t)$ as given in Proposition 1 by $\mathbf{V}(t)$ so that

$$B_j(t) = X_j\left(V_j(t)\right) \quad j = 1, ..., n.$$

Unlike Lévy processes, the process $\mathbf{B}(t)$ generates a stochastic conditional variance/covariance matrix.

The corresponding characteristic function of the margin process is given by

$$\phi_{Bj}(u;t) = \mathbb{E}\left[\mathbb{E}\left(e^{iuX_j(V_j(t))} \middle| V_j(t)\right)\right] \quad u \in \mathbb{R}, \quad j = 1, ..., n; \tag{4.1}$$

if $V_j(t)$ is independent of $X_j(t)$ for $j=1,...,n$, (4.1) reduces to

$$\phi_{Bj}(u;t)=\phi_{Vj}\left(-i\left(\varphi_{Yj}(u)+\varphi_Z(ua_j)\right);t\right) \quad j=1,...,n. \tag{4.2}$$

The generalization of the previous result to the case in which $X_j(t)$ and $V_j(t)$ are correlated (to capture the so called leverage effect) can be obtained using the leverage-neutral measure of Carr and Wu (2004).

For an illustration, we consider the simple case of a multivariate VG process (as given in section 2.2) time changed by an independent integrated CIR process (as in Carr et al., 2003), so that

$$V_j(t)=b_j\int_0^t v(s)ds \quad j=1,...,n,$$

where $b_j>0$ for $j=1,...,n$, the (common) variance rate $v(t)$ satisfies the stochastic differential equation

$$dv(t)=\kappa\left(\eta-v(t)\right)dt+\lambda\sqrt{v(t)}d\bar{W}(t),$$

and $\bar{W}(t)$ is a standard Brownian motion independent of the base process $\mathbf{X}(t)$. The characteristic function of $V_j(t)$ is well known from standard results on affine processes (see, e.g., Filipović, 2009); therefore (4.2) reads

$$\phi_{Bj}(u;t)=e^{\Phi_j(u,t)+\Psi_j(u,t)v(0)} \quad j=1,...,n,$$

with

$$\begin{aligned}
\Phi_j(u,t) &= \frac{2\kappa\eta}{\lambda^2}\ln\left(\frac{2\zeta_j(u)e^{\frac{\zeta_j(u)+\kappa}{2}t}}{\left(\zeta_j(u)+\kappa\right)\left(e^{\zeta_j(u)t}-1\right)+2\zeta_j(u)}\right)\\
\Psi_j(u,t) &= \frac{2\xi_j(u)\left(e^{\zeta_j(u)t}-1\right)}{\left(\zeta_j(u)+\kappa\right)\left(e^{\zeta_j(u)t}-1\right)+2\zeta_j(u)}\\
\zeta_j(u) &= \sqrt{\kappa_j^2-2\lambda^2\xi_j(u)}\\
\xi_j(u) &= -\frac{b_j}{\nu_j}\ln\left(1-iu\beta_j\nu_j+u^2\frac{\gamma_j^2}{2}\nu_j\right)-\frac{b_j}{\nu_Z}\ln\left(1-iua_j\beta_Z\nu_Z+u^2\frac{a_j^2\gamma_Z^2}{2}\nu_Z\right).
\end{aligned}$$

We note the limited dependence structure offered by the proposed construction due to the common time change applied to the base Lévy process; a full, richer construction of multivariate time changed Lévy processes is left to future research.

The parameters of the multivariate VG-CIR model calibrated to the market data described in section 3.1, under the assumption of a risk neutral dynamic of the stock price

$$S_j(t)=S_j(0)e^{(r-q)t}\frac{e^{B_j(t)}}{\mathbb{E}\left(e^{B_j(t)}\right)},$$

are reported in Table 7. For illustration purposes, we only consider the valuation date as of 27/02/2009. The Table reports the error in fitting the correlation matrix as well as the error in

Table 7 Calibration of the multivariate VG-CIR model.

VG-CIR MODEL								
MARGINS					MOMENT MATCHING ERROR			
		F	ABT	BAX		F	ABT	BAX
VG	θ	-3.1330	-0.7165	-0.73366	$\mathbb{E}X\,(1)$	0.00E+00	0.00E+00	0.00E+00
	σ	1.0542	0.3296	0.3513	$stdX\,(1)$	-4.31E-02	-2.94E-02	-1.58E-02
	k	0.0314	0.1836	0.0927	$\gamma_1\,(1)$	7.15E-03	-7.07E-01	-9.79E-03
	b	1.0000	0.2351	0.2220	$\gamma_2\,(1)$	-1.98E-02	-7.03E-01	-2.80E-01
CIR	λ	0.8333	0.4040	0.3926				
	κ	1.0993	1.0993	1.0993				
	η	1.1275	0.2651	0.2503	CALIBRATION ERROR			
	RMSE	7.63E-03	1.33E-01	7.39E-02	RMSE	-8.92E-11	8.22E-13	5.15E-11
	(w)RMSE	8.53E-04	8.18E-03	3.66E-03	(w)RMSE	-7.55E-10	4.93E-09	2.28E-09
IDIOSYNCRATIC PART					CORRELATION ERROR			
	β	-1.8899	0.0962	-0.1144	F	-		
	γ	0.9680	0.1849	0.2825	ABT	-2.73E-07	-	
	ν	0.0372	2.2361	0.1727	BAX	-2.18E-07	-9.97E-07	-
	a	1.1932	0.7801	0.5972				
SYSTEMIC PART								
	β_Z	-1.0418						
	γ_Z	0.3498						
	ν_Z	0.2000						

Note. Valuation date: 27/02/2009. Parameters of marginal distributions $(\theta_j, \sigma_j, k_j, \lambda_j, \kappa, \eta_j)$ obtained by direct calibration to market prices (note: $\lambda_j = \lambda\sqrt{b_j}$, $\eta_j = b_j\eta$, where λ, η are the parameters of the common time change). Remaining parameters (idiosyncratic and systematic components of the VG process) obtained by fitting the correlation matrix subject to relevant convolution conditions. Moment matching error: difference between the exact moments provided in Appendix B (calculated using the parameters of the marginal process) and the moments reconstructed using equations (B.3)-(B.6) in conjunction with equation (2.2). Correlation error: difference between the model and the sample correlation. Calibration error: difference between the errors of the calibration of the margin process and the linear combination to the market prices.

reproducing the original option prices by the multivariate VG-CIR model. Comparison with Table 2 shows the improved performance of the time changed VG construction due to the additional stochastic volatility features. Further evidence is provided in Figure 2, where we plot the implied volatilities generated by the multidimensional VG-CIR process and compare them with the ones obtained previously from the multivariate VG model. In particular, we note that the implied volatility induced by the VG-CIR construction provides a better fit especially for the more liquid contracts, as expected (see, e.g., Carr et al., 2003 and Huang and Wu, 2004).

The multivariate construction for time changed Lévy processes introduced in this section can be further improved to a setting in which stochastic volatility can be generated separately from the diffusion and the jump component of $\mathbf{X}(t)$, by applying individual time changes as in Huang and Wu (2004). Hence, the multidimensional model proposed in this section could be considered as an alternative to the Wishart processes approach introduced for example by Gourieroux (2006), Da Fonseca et al. (2007) (2007) and further extended by Leippold and Trojani (2008).

5. Conclusions

In this note we studied an alternative construction of multivariate Lévy processes which keeps the appealing properties of the approaches existing in the literature and, at the same time, addresses their limitations. The relevance of this topic arises from the increasing need of more sophisticated models as highlighted by the recent financial crisis. The proposed model could also be used as a platform to construct multivariate time changed Lévy processes, allowing for a richer stochastic volatility structure.

The empirical analysis presented in this paper shows that our approach is flexible enough to accommodate the full range of possible linear dependence, from negative to positive correlation, from complete linear dependence to independence, but, at the same time, it is relatively parsimonious in terms of number of parameters involved, as this grows linearly with the number of names in the basket.

Finally, in the applications discussed in this paper, we assume that the margins and the overall multivariate process follow the same distribution; this is necessary in order to maintain the calibration of the margin processes separated from the fitting of the correlation structure. However, Proposition 1 holds for any Lévy process and therefore it allows to relax this assumption.

Acknowledgments

The authors would like to thank Damiano Brigo, Carole Bernard, Elisa Luciano, Pietro Millossovich, Lorenzo Trapani and Steven Vanduffel for providing valuable comments, criticisms and further references. Further thanks go to Frederik Ulmer and Yanmin Li for helpful discussions and suggestions on model calibration issues. A previous version of this paper was circulated under the title 'A note on multivariate asset models using Lévy processes'. The paper in its present format was presented at the IRMC 2010 Conference, the EUROFIDAI/AFFI 8th International Paris Finance Meeting, the Risk and Stochastics Day 2011 at LSE, the 5th International Conference MAF2012 and the 2nd International Conference of the Financial Engineering and Banking Society (FEBS2012). We thank all the participants for their helpful feedback. The views expressed in this paper are those of the authors and do not necessarily reflect those of Mitsubishi UFJ Securities Int. PLC. Usual caveat applies.

Appendix A: Proof of statements in Section 2

A.1. Proof of Corollary 1

Proof.

(i) It follows from direct differentiation of the characteristic function of the components of $\mathbf{X}(t)$ and from the fact that Lévy processes have an infinitely divisible distribution.

(ii) It follows from the assumption that the process $\mathbf{Y}(t)$ has independent components and it is independent of $Z(t)$.

■

A.2. Proof of Corollary 2

Proof. Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be increasing functions for which the covariances are defined. Then

$$\mathbb{C}ov\left(f\left(\mathbf{X}\left(t\right)\right), g\left(\mathbf{X}\left(t\right)\right)\middle| Z\right) \geq 0$$

due to the association of $\mathbf{Y}\left(t\right)$. Further, for any increasing function f,

$$\mathbb{E}\left(f\left(\mathbf{X}\left(t\right)\right)\middle| Z = z\right) = \mathbb{E}\left(f\left(\mathbf{Y}\left(t\right) + \mathbf{a}z\right)\right)$$

is an increasing function of z if $a_j > 0$ for $j = 1, 2, ..., n$, which implies

$$\mathbb{C}ov\left(\mathbb{E}\left(f\left(\mathbf{X}\left(t\right)\right)\middle| Z\right), \mathbb{E}\left(g\left(\mathbf{X}\left(t\right)\right)\middle| Z\right)\right) \geq 0$$

as $Z(t)$ is positive associated. On the other hand, if $a_j < 0$ for all $j = 1, 2, ..., n$, we note

$$\mathbb{C}ov\left(\mathbb{E}\left(f\left(\mathbf{X}\left(t\right)\right)\middle| Z\right), \mathbb{E}\left(g\left(\mathbf{X}\left(t\right)\right)\middle| Z\right)\right) = \mathbb{C}ov\left(-\mathbb{E}\left(f\left(\mathbf{X}\left(t\right)\right)\middle| Z\right), -\mathbb{E}\left(g\left(\mathbf{X}\left(t\right)\right)\middle| Z\right)\right)$$

which is non-negative as well. Therefore,

$$\begin{aligned}\mathbb{C}ov\left(f\left(\mathbf{X}\left(t\right)\right), g\left(\mathbf{X}\left(t\right)\right)\right) &= \mathbb{E}\left(\mathbb{C}ov\left(f\left(\mathbf{X}\left(t\right)\right), g\left(\mathbf{X}\left(t\right)\right)\middle| Z\right)\right) \\ &\quad + \mathbb{C}ov\left(\mathbb{E}\left(f\left(\mathbf{X}\left(t\right)\right)\middle| Z\right), \mathbb{E}\left(g\left(\mathbf{X}\left(t\right)\right)\middle| Z\right)\right) \geq 0\end{aligned}$$

as required (see for example Müller and Stoyan, 2002). ■

Appendix B: Time changed Lévy processes

Time changed Lévy processes are obtained by observing a Lévy process $X(t)$ on a time scale governed by a non-negative, non-decreasing stochastic process $V(t)$. $X(t)$ is the base process, $V(t)$ is the time change, or stochastic clock, and the resulting process is $B(t) = X\left(V(t)\right)$. The corresponding characteristic function is given by

$$\phi_B(u;t) = \mathbb{E}\left[\mathbb{E}\left(e^{iuX(V(t))}\middle| V(t)\right)\right] \quad u \in \mathbb{R}, \tag{B.1}$$

which, under the assumption of a stochastic clock independent of the base process, can be rewritten as

$$\phi_B(u;t) = \phi_V\left(-i\left(\varphi_X(u)\right);t\right) \tag{B.2}$$

In this case, it follows by direct differentiation of the (log of the) characteristic function of $B(t)$ that

$$\mathbb{E}B(t) = \mathbb{E}(X(1))\,\mathbb{E}(V(t)), \tag{B.3}$$

$$\mathbb{V}ar(B(t)) = \mathbb{V}ar(X(1))\,\mathbb{E}(V(t)) + \mathbb{E}^2(X(1))\,\mathbb{V}ar(V(t)), \tag{B.4}$$

$$c_3(B(t)) = c_3(X(1))\,\mathbb{E}(V(t)) + 3\mathbb{E}(X(1))\,\mathbb{V}ar(X(1))\,\mathbb{V}ar(V(t)) + \mathbb{E}^3(X(1))\,c_3(V(t)), \tag{B.5}$$

$$\begin{aligned} c_4(B(t)) &= c_4(X(1))\mathbb{E}(V(t)) + 4c_3(X(1))\mathbb{E}(X(1))\,\mathbb{V}ar(V(t)) + 3\mathbb{V}ar^2(X(1))\,\mathbb{V}ar(V(t)) \\ &\quad + 6\mathbb{E}^2(X(1))\,\mathbb{V}ar(X(1))\,c_3(V(t)) + \mathbb{E}^4(X(1))\,c_4(V(t)), \end{aligned} \tag{B.6}$$

from which the indices of skewness, $\gamma_1(t)$, and excess kurtosis, $\gamma_2(t)$, follow.

In the special case in which the base process is a Brownian motion with drift of the form $X(t) = \theta t + \sigma W(t)$ for $\theta \in \mathbb{R}, \sigma > 0$, then (B.3)-(B.6) reduce to (see, for example, Ané and Geman, 2000)

$$\mathbb{E}B(t) = \theta\mathbb{E}(V(t)), \tag{B.7}$$

$$\mathbb{V}ar(B(t)) = \sigma^2\mathbb{E}(V(t)) + \theta^2\mathbb{V}ar(V(t)), \tag{B.8}$$

$$c_3(B(t)) = 3\theta\sigma^2\mathbb{V}ar(V(t)) + \theta^3 c_3(V(t)), \tag{B.9}$$

$$c_4(B(t)) = \theta^4 c_4(V(t)) + 6\theta^2\sigma^2 c_3(V(t)) + 3\sigma^4\mathbb{V}ar(V(t)). \tag{B.10}$$

Other Lévy processes with known time changed Brownian motion representation are the CGMY (Carr et al., 2002) and the Meixner process (see Madan and Yor, 2008).

Appendix C: Cumulants of a JD process

By differentiation of the characteristic exponent, it follows

$$\mathbb{E}X(t) = (\mu + \lambda\mathbb{E}(\xi))\,t, \tag{C.1}$$

$$\mathbb{V}ar(X(t)) = \left(\sigma^2 + \lambda\mathbb{E}\left(\xi^2\right)\right)t, \tag{C.2}$$

$$c_3(X(t)) = \lambda\mathbb{E}\left(\xi^3\right)t, \tag{C.3}$$

$$c_4(X(t)) = \lambda\mathbb{E}\left(\xi^4\right)t. \tag{C.4}$$

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