Asset modeling, stochastic volatility and stochastic correlation

"Everything that relies on correlation is charlatanism"
(Nassim Taleb)

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Abstract

Financial correlation modeling is challenging, since financial correlations behave volatile and erratic. However, we build a simple stochastic volatility – stochastic correlation model, which fits empirical correlation properties well. In addition, the modeling of asset prices and the modeling of correlation is currently not unified in a coherent model. We integrate our model into the standard geometric Brownian motion (GBM) to create a unified asset price – asset correlation model. This model has a CAPM interpretation and replicates asset prices in reality significantly better that the standard GBM. We also apply a conditionally independent (CID) correlation approach between individual stocks in a portfolio approach. This portfolio approach improves asset modeling further.

Keywords: Asset modeling, stochastic volatility, stochastic correlation, Heston model

JEL Classification: C12, C32

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1. Introduction and Motivation

Asset prices are typically modeled with the Geometric Brownian motion (GBM) of the form

$$d \ln S(t) = \mu\, dt + \sigma\, dw(t) \qquad (1)$$

where S(t) is the asset price, μ is the drift of S, σ is the volatility of S, and dw(t) is a standard Brownian motion, i.e. $dw(t) = \varepsilon_t \sqrt{dt}$, ε_t is i.i.d., in particular ε_t is a random drawing from a standardized normal distribution n~(0,1).

Numerous extensions of the GBM exist. Merton (1976) adds jumps to the GBM and shows that if the logarithm of the percentage jump is normally distributed, a closed form solution for European style options exists. Cox and Ross (1976) create the constant elasticity of variance (CEV) model, where an exponential parameter α added to the asset price. The value of α determines the dependence between asset price and volatility. In a pure jump extension, Madan et al create a variance-gamma approach, which create heavier tails and provides semi-analytic expressions for European style options.

Heston (1993) in a seminal model correlated the GBM with stochastic volatility. Numerous applications and extensions of the Heston model exist as for example Zhou (2001), Hagan et al (2002), Brigo and Pallacinini (2008), and Langnau (2009).

While stochastic volatility has been applied to the GBM, stochastic correlation is a fairly new field, which still awaits integration into a stock price process as the GBM. Even deterministic financial correlation approaches are not unified with a stock price model. Let's briefly review the most popular correlation approaches in finance.

Some authors as Das et al (2006) or Fitch (2006) apply the Pearson correlation coefficient. However, the limitations of Pearson correlation approach

in finance are evident. First, linear dependencies as assessed in the Pearson approach do not appear often in finance. In addition, zero Pearson correlation does not necessarily mean independence. This is because only the two first moments are considered. For example, $Y = X^2 \{y \neq 0\}$ will lead to a Pearson correlation coefficient of 0, which is arguably misleading. Furthermore, linear correlation measures are only natural dependence measures if the joint distribution of the variables is elliptical. However, only few distributions such as the multivariate normal distribution and the multivariate student-t distribution are special cases of elliptical distributions, for which linear correlation measure can be meaningfully interpreted.

A further popular correlation measure, mainly applied to default correlation, is the binomial correlation approach of Lucas (1995). However, the binomial correlation approach is a limiting case of the Pearson correlation approach. As a consequence, the significant shortcomings of the Pearson correlation approach for financial modeling apply also to the binomial correlation model.

One of the most widely applied correlation approaches in finance was generated by Steven Heston in 1993. Heston correlates the Brownian motion of a stock price of equation (1) with the Brownian motion of a mean reverting stochastic volatility process. We will apply the Heston approach in our paper to correlate stochastic volatility and stochastic correlation.

Further popular correlation approaches applied in finance are Copula correlations introduced by Sklar (1959) and Li (2000). A factorization of the copula approach leads to conditionally independent (CID) correlation modeling, which was introduced by Vasicek (1987), and extended by Hull et al (2005) and Burtschell et al (2007). We will apply the CID correlation approach in a portfolio extension of our model.

Further correlation models are contagion modeling, introduced by Davis and Lo (2001) and Jarrow and Yu (2001). A fairly new class of correlation models are top down approaches, introduced by Schönbucher (2006), Hurd Kuznetsov (2006a and 2006b), Giesecke and Tomecek (2005) and Giesecke et al (2009).

For an overview of existing correlation approaches applied in finance, see Albanese et al (2010).

Importantly, the geometric Brownian motion and the existing correlation approaches are currently not unified in a consistent model. Rather, the correlations between one or more assets are exogenously derived and then ad hoc added to the Brownian motion. This is mathematically and conceptually unsatisfying.

To the best of our knowledge, there are only two papers that apply stochastic correlation to the GBM. One is Hull et al (2005). They model the asset process with a GBM and then sample the correlation from a beta distribution. However, their stochastic correlation is exogenously derived and does not follow a time dependent stochastic process. The second paper is Ma (2009). Here stochastic volatility and stochastic correlation processes are applied to price exchange rate options. However, volatility and correlation are assumed independent.

Emmerich (2006) discusses mathematical properties of a stochastic correlation process. Duellman et al (2008) model stochastic correlation with a Vasicek process. However, both papers do not combine the correlation process with the asset process.

Several other papers mention the term 'stochastic correlation' as Burtschell et al (2005), Burachi et al (2006) or Fonesca et al (2008). However, in these papers correlation is stochastically sampled from a distribution or is inferred from a historical price correlation matrix. The correlation does not follow a time dependent stochastic process with drift and noise.

Our paper has three main contributions. First, we build a stochastic volatility – stochastic correlation model, which fits real world volatility – correlation relationships well. Second, we integrate our model into the geometric Brownian motion (GBM). This new GBM has a CAPM interpretation and improves the modeling of stock prices significantly. Third, we apply a portfolio approach in

which we correlate individual stocks with the conditionally independent (CID) correlation model. This portfolio approach results in a better fit of real world data than the modeling of individual stocks.

The remaining paper is structured as follows: In section 2 we build a combined stochastic asset volatility – stochastic asset correlation model. In section 3 we apply the model to extend the GBM. In section 4 we show that our model can replicate the empirical asset process better than the standard GBM. Section 5 concludes.

2. The Model

We suggest a new, simple approach to simultaneously model stochastic volatility and stochastic correlation. In particular, we model

$$d\sigma = (a_\sigma - b_\sigma)dt + \upsilon_\sigma \sqrt{\sigma}\, dw_1 \qquad (2)$$

$$d\rho = (a_\rho - b_\rho)dt + \upsilon_\rho \sqrt{1-\rho^2}\, dw_2 \qquad (3)$$

$$dw_1 = \rho_w dw_2 + \sqrt{1-\rho_w^2}\, dw_3 \qquad (4)$$

Equation (2) models stochastic volatility of the market, represented by the S&P 500, with the Cox, Ingersoll, Ross (CIR) model. Hence we have σ : implied Volatility (VIX) of the S&P 500, a_σ : mean reversion of σ, b_σ : long term mean of σ, v_σ : volatility of σ.

Equation (3) models stochastic correlation between an individual stock and the market with a modified Jacobi process. Hence a_ρ : mean reversion of ρ, b_ρ : long term mean of ρ, v_ρ : volatility of ρ.

Equation (4) correlates the Brownian motion dw_1 of the stochastic volatility process and the Brownian motion dw_2 of the stochastic correlation process with the Heston (1993) model. Hence dw_2 and dw_3 are independent Brownian motions

and each Brownian motion is a Markov process, i.e. dw(t) and dw(t') are independent, t ≠ t'.

Real world fit

From empirical data, we derive historical volatility and correlation properties as seen in Figure 1:

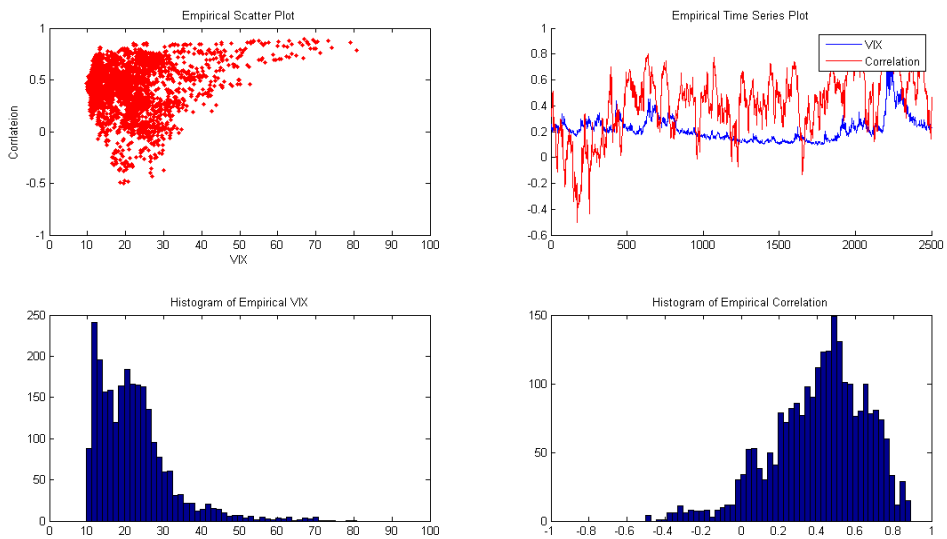

Figure 1: Empirical relationship between implied volatility (VIX) of the S&P, and the correlation between Chevron Corp and the S&P (top-left).Time series plot of the empirical VIX and empirical correlation between CVX and the S&P (top-right). Histogram of VIX (bottom-left). Histogram of the correlation coefficient (bottom-right).

We expect that our model produces a similar positive, 'triangular' relationship between volatility σ and correlation ρ as in Figure 1 top left, since if ρ_w in equation (4) is positive, we have a positive dependency between σ and ρ. In addition, the volatility of correlation v_ρ decreases if volatility increases due to the term $\upsilon_\rho\sqrt{1-\rho^2}$ in equation (3).

We calibrate the model of equations (2) to (4) using standard Maximum Likelihood techniques. In particular, we use 100,000 time steps for each equation

with dt = 0.1. To stabilize the data, we repeat this process 100 times and average the data. We find that our model replicates real-world volatility–correlation properties of Figure 1 well:

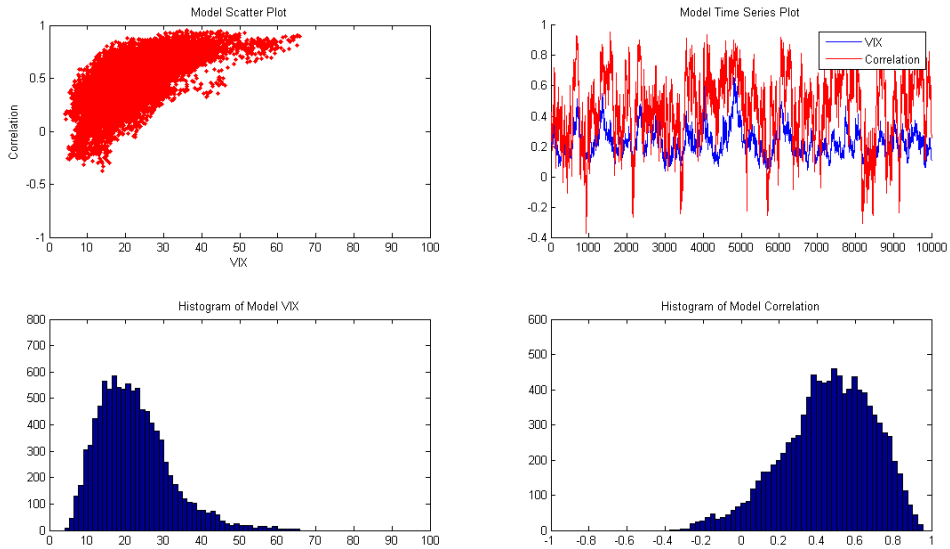

Figure 2: Simulation results of the model of equations (2) to (4) with parameter values a_σ=0.0104, b_σ=0.0140, v_σ=0.0075, a_ρ=0.0158, b_ρ=0.4249, v_ρ=0.0504, and ρ_w=0.7

3. Application of the model

We now apply our model of equations (2) to (4) to improve the geometric Brownian motion (GBM) of equation (1). Dropping the argument t to simplify notation, we model

$$d \ln S_i = \mu_i \, dt + \sigma_i \, dw_i + \beta_i \, \rho \, \sigma \, dw \qquad (5)$$

where S_i is the asset price of entity i, μ_i is the drift of S_i, σ_i is the volatility of S_i, and β_i is a positive constant. ρ is the correlation between an individual stock and the market, represented by the S&P 500. ρ is modeled as a stochastic process by equation (3). σ is the volatility of the market, represented by the VIX of the S&P

500, which is modeled by equation (2). The Brownian motions of ρ and σ are correlated via equation (4). dw is the Brownian motion of the market component.

Equation (5) has a CAPM interpretation. The first two terms on the right side of equation (5) represent the idiosyncratic stock component. The term σ dw represents the systematic market risk factor, which is shared by all stocks. The impact magnitude of systematic component on the stock is β_i ρ.

4. Results

We again use Maximum Likelihood techniques to calibrate the Geometric Brownian motion (GBM) of equation (1), and our approach of equations (2) to (5). In our approach, we model the variables ρ and σ in equation (5) with equations (2) to (4). We then optimize the critical parameters σ_i and β_i, so that the absolute difference of standard deviation, skewness and kurtosis between the real-world distribution and our distribution of equation (5) is minimized.

We derive the distributions for Coca Cola Corporation (KO) as seen in Fi

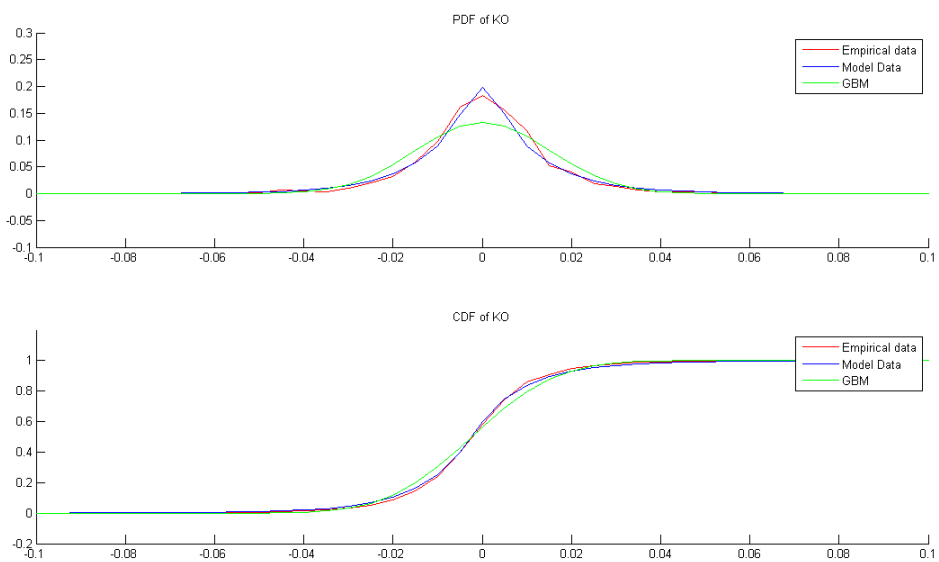

Figure 3: PDF and CDF for Coca Cola Corporation (KO). Model data of equations (2) to (5) is derived with parameter values a_σ=0.0104, b_σ=0.0140, v_σ=0.0075, a_ρ=0.0245, b_ρ=0.3136, v_ρ=0.0530, ρ_w=0.7, μ_i = 8.068x10^{-5}, σ_i = 0.034, and β_i = 2.1119. The time unit is daily.

From Figure 3 we observe that our model of equations (2) - (5) replicates the market distribution of KO better than the standard GBM of equation (1). This is verified by standard statistics. The Chi-square goodness-of-fit test shows a p-value of 0.8164 (chi^2=5.986) between our model distribution and the empirical distribution, while the p-value is 0.054 (chi^2=19.411) between the GBM-normal distribution and the empirical distribution. Our model gives similar results for other stocks that we tested.

In addition, our model catches fat tails as seen in Figure 4.

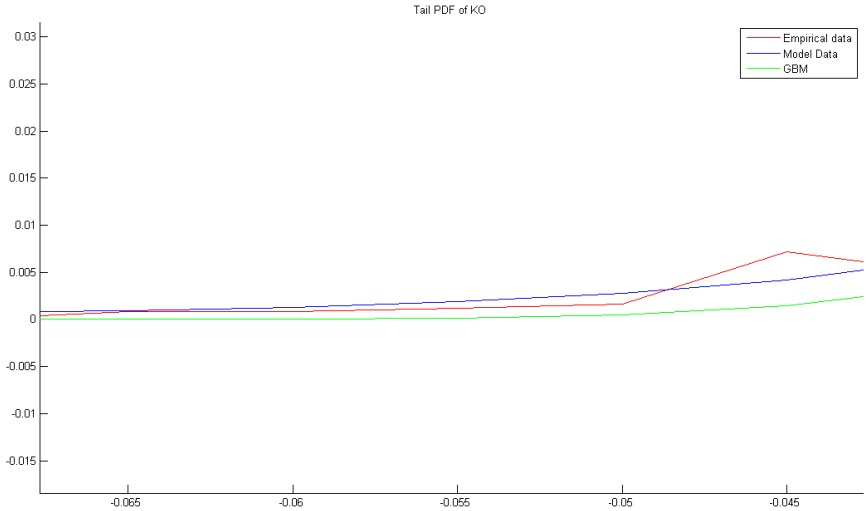

Figure 4: Tail distribution of Coca Cola Corporation (KO). Model data derived by equations (2) to (5) with parameter values as in Figure 3.

Portfolio approach

As discussed, we correlate stochastic volatility of the market σ and stochastic correlation ρ between the market and an individual stock with equations (2) to (4).

In a portfolio approach we additionally model the correlation between individual stocks. Correlating multiple assets can be conveniently achieved with the conditionally independent (CID) correlation model, introduced by Vasicek (1987), and extended by Hull et al (2005) and Burtschell et al (2007). Hence, we indirectly condition the individual stock prices S_i, i=1,..,n in the portfolio on a common market factor σ dw as seen in equation (5). The impact of the common market factor on a specific stock i, is β_i ρ.

The result of the portfolio approach is displayed in Figure 5. We model the Dow components ATT (T), Coca Cola (KO), Disney (DIS), and Pfizer (PFE) and Verizon (VZ), each with equations (2) to (5) and average the outcome.

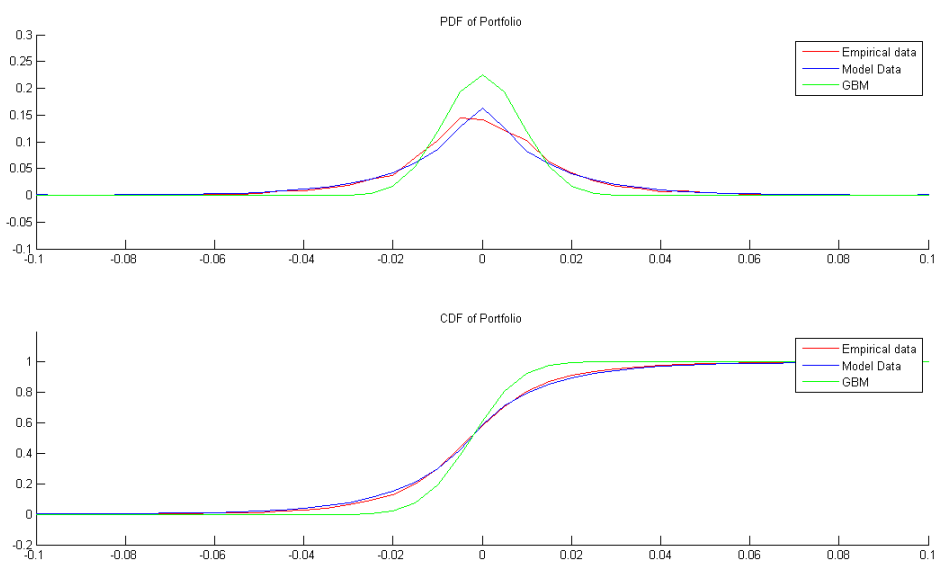

Figure 5: PDF and CDF of a portfolio of 5 equally weighted Dow components. Parameter values for the volatility process, correlation process and the individual stock processes are in the Appendix.

Comparing Figures 3 and 5, we observe that the portfolio approach in Figure 5 results in a slightly better fit of the empirical data than modeling an individual stock in Figure 3. This is confirmed by standard statistics. The p-value increases from 0.8164 (chi^2=5.986) to 0.9605 (chi^2=5.560). When testing different portfolios and comparing the result to modeling individual stocks, we get similar results. The reasons for the better results of the portfolio approach are twofold. First, the correlation between individual stocks is applied in the portfolio approach. Second, differences of variance, skew and kurtosis between the individually modeled stocks and the empirical data are smoothed in the portfolio approach.

Further research

The basic model presented here can be extended in numerous ways. We can add jumps to the GBM (Merton 1976), extent the GBM to the CEV (constant elasticity of variance) model (Cox and Ross 1976), or test a pure jump model as the Variance-Gamma model (Madan et al 1998). We can test if other variables as bonds, commodities, exchange rates, credit spreads, economic variables, real estate values, weather data as temperature or precipitation etc. can be replicated with our suggested model or variations of the model.

In addition, we could make the parameter ρ_w in equation (4), which correlates stochastic volatility and stochastic correlation, stochastic and dependent on other variables. The same applies to the other parameters as μ_1 and σ_1. We could also add a drift term to the market factor in equation (5).

We could also use a different model to generate the volatility process and correlation process as the ARCH or GARCH model (Engle 1982, Bollerslev 1986) and then correlate these processes.

Regarding the application of the model, we may test if the model replicates realistic implied volatility smiles.

We also found a semi-quadratic relationship between asset return and asset correlation, i.e. correlation is high for negative returns and correlation is high, but to a lesser extent, for positive returns. However, this relationship

appears to be quite instable.

5. Concluding Summary

This paper has three main contributions. First, we first build a simple but rigorous stochastic volatility – stochastic correlation model, in which the Brownian motions of the volatility process and the correlation process are correlated with the seminal Heston 1993 approach. This model fits empirical correlation properties well.

Second, asset correlations are currently modeled exogenously and then ad hoc assigned to an asset price process as the Geometric Brownian motion (GBM). This is conceptually and mathematically unsatisfying. We apply our model to build a unified asset price - asset correlation model. This model has a CAPM interpretation and replicates asset prices in reality significantly better that the standard GBM. The model also captures fat tails well.

Third, we apply a conditionally independent (CID) correlation approach between individual stocks in a portfolio approach. This portfolio approach improves asset modeling further.

The basic model presented here can be extended in numerous ways.

Appendix:

The parameter values for the model

$$d\sigma = (a_\sigma - b_\sigma)dt + \upsilon_\sigma \sqrt{\sigma}\, dw_1 \qquad (2)$$

$$d\rho = (a_\rho - b_\rho)dt + \upsilon_\rho \sqrt{1-\rho^2}\, dw_2 \qquad (3)$$

$$dw_1 = \rho_w dw_2 + \sqrt{1-\rho_w^2}\, dw_3 \qquad (4)$$

$$d \ln S_i = \mu_i\, dt + \sigma_i\, dw_i + \beta_i\, \rho\, \sigma\, dw \qquad (5)$$

for the individual stock processes S_i in the portfolio approach discussed in section 4, displayed in Figure 5, are

DIS: $a_\sigma = 0.0104$, $b_\sigma = 0.0140$, $v_\sigma = 0.0075$, $a_\rho = 0.0265$, $b_\rho = 0.4803$, $v_\rho = 0.0533$, $\rho_w = 0.7$, $\mu_i = 5.9791\text{x}10^{-5}$, $\sigma_i = 0.0027$, and $\beta_i = 2.7308$.

KO: $a_\sigma = 0.0104$, $b_\sigma = 0.0140$, $v_\sigma = 0.0075$, $a_\rho = 0.0245$, $b_\rho = 0.3136$, $v_\rho = 0.0530$, $\rho_w = 0.7$, $\mu_i = 8.068\text{x}10^{-5}$, $\sigma_i = 0.034$, and $\beta_i = 2.1119$.

PFE: $a_\sigma = 0.0104$, $b_\sigma = 0.0140$, $v_\sigma = 0.0075$, $a_\rho = 0.0262$, $b_\rho = 0.3876$, $v_\rho = 0.0528$, $\rho_w = 0.7$, $\mu_i = -1.0949\text{x}10^{-4}$, $\sigma_i = 0.023$, and $\beta_i = 2.6527$.

T: $a_\sigma = 0.0104$, $b_\sigma = 0.0140$, $v_\sigma = 0.0075$, $a_\rho = 0.0231$, $b_\rho = 0.3850$, $v_\rho = 0.0491$, $\rho_w = 0.7$, $\mu_i = -5.6201\text{x}10^{-4}$, $\sigma_i = 0.031$, and $\beta_i = 2.3344$.

VZ: $a_\sigma = 0.0104$, $b_\sigma = 0.0140$, $v_\sigma = 0.0075$, $a_\rho = 0.0262$, $b_\rho = 0.3876$, $v_\rho = 0.0528$, $\rho_w = 0.7$, $\mu_i = -1.0949\text{x}10^{-4}$, $\sigma_i = 0.023$, and $\beta_i = 2.6527$.

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